LaPlace Transforms in Design and Analysis of Circuits

Part 1: Basic Transforms

Course No: E04-015

Credit: 4 PDH

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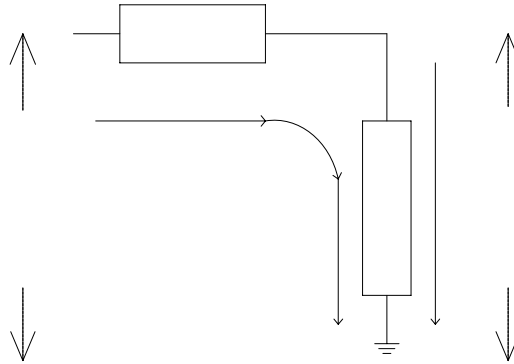# LaPlace Transforms in Design and Analysis of Circuits©

Part 1 - Basic Transforms

by Tom Bertenshaw

Why Use the LaPlace Transform??

In a short synopsis; using the LaPlace transform method of solving circuit differential equations allows the building of simple <u>algebraic</u> transfer functions that mathematically model the actual circuit; provides a quick method for calculating transfer function amplitude and phase as a function of frequency; and creates a foundation for the rapid calculating and graphing of circuit loop behavior with respect to stability. Summing the above, the use of transforms provides a simple procedure for performing an essential engineering function; i.e., ***predict circuit output as a function of input***. We will get to all of these as a matter of course, but first comes the fundamentals.

We are all familiar with the concept of calculating the output by using a voltage division network, i.e.,

because the circuit is *series*, $I_t = I_o$, then:

$$\frac{V_{in}}{z_1 + z_2} = \frac{V_o}{z_2} \text{ or}$$

$$V_o = \left(\frac{z_2}{z_1 + z_2} \right) * V_{in}$$

which is a very handy and simple way of predicting the output for any given input. It is, in fact, a very simple transfer function. Be aware that we are slanting the whole further development of analysis and design around the <u>transfer function technique</u> as illustrated in the simple example above.

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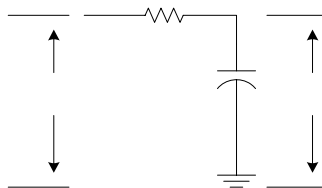Notice that the equation has the general form of

$$out = (Xfer) * in$$

where the term *Xfer* means "transfer function". Re-arranging this equation isolates and defines the transfer function as

$$\frac{out}{in} = Xfer$$

Transfer functions have the innate ability to <u>allow</u> <u>prediction</u> <u>of</u> <u>output</u> <u>as</u> <u>a</u> <u>function</u> <u>of</u> <u>input</u>, and as such are extremely valuable engineering tools. Getting there when the circuit equations are integral/differential can be cumbersome, and error fraught. Nevertheless circuit equations are generally integral/differential that can be masked and "worked around" by LaPlace's technique.

A Series Circuit

Consider a low-pass filter;

The characteristic equation describing the voltage/current relationship for a **resistor** is:

$$V = I * R \text{ or } v = i \times r \text{ (lower case denotes time changing variables)}$$

whereas for a **capacitor** the relationship is:

$$i = C\frac{dv}{dt}$$

Because a capacitor is an open to DC, the vast majority of circuit problems in design or analysis that include capacitors occur in alternating or changing current situations. i.e., there is a $\frac{dv}{dt}$.

Forming the ratio V_o/V_{in} for an RC network is not as straightforward as developing the impedance ratio as in the first case mentioned for the general, or frequency independent, case. However a general method of solving the problem follows, and that is how we will proceed.

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Observing that the circuit is a series circuit, and realizing that the current through the resistor *MUST* equal the current through the capacitor, we can write:

$$i_r = i_C \quad \text{or}$$

$$\frac{v_{in} - v_c}{R} = C \frac{dv_c}{dt}$$

where v_c and V_o represent the same voltage. In other words, $V_o = v_c$; and v_{in} is whatever the case may be: DC, changing DC or AC. The notation for voltage is changed to denote that it can be time variable.

After suitable rearranging we get:

$$dv_c + \frac{v_c}{RC} dt = \frac{V_{in}}{RC} dt$$

This relationship has an integrating factor of:

$$e^{\frac{1}{RC}\int dt} \quad \text{or } e^{\frac{t}{RC}}.$$

After both sides are multiplied by the integrating factor, the equation becomes:

$$d\left(v_c e^{\frac{t}{RC}}\right) + e^{\frac{t}{RC}} \frac{v_c}{RC} dt = \frac{V_{in} e^{\frac{t}{RC}}}{RC} dt$$

Integrating both sides gives us:

$$v_c e^{\frac{t}{RC}} = \frac{v_{in}}{RC} e^{\frac{t}{RC}} + c_{nst} \quad \text{where } c_{nst} \text{ is the constant of integration.}$$

Isolating the dependent variable (v_c) the relationship becomes:

$$v_c = \frac{v_{in}}{RC} + c_{nst} e^{-\frac{t}{RC}}$$

at $t = 0$; $c_{nst} = -\frac{v_{in}}{RC}$; therefore the complete solution is:

$$v_c = \frac{v_{in}}{RC}\left(1 - e^{-\frac{t}{RC}}\right)$$

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After learning LaPlace Transform pairs and their applications, and having appealed to the use of the LaPlace Transform instead of using Ordinary Differential Equations, the process devolves to simple Algebra.

$\dfrac{v_{in} - v_c}{R} = C\dfrac{dv_c}{dt}$ becomes by direct application:

$$\frac{v_{in}(s) - v_c(s)}{R} = sCv_c(s) \text{ (Step 1)}$$

rearranging yields:

$$v_c(s) = \frac{v_{in}(s)}{RC\left(s + \dfrac{1}{RC}\right)} \text{ (Step 2)}$$

which directly inverts to:

$$v_c = \frac{v_{in}}{RC}\left(1 - e^{-\frac{t}{RC}}\right) \text{ (Step 3)}$$

a far simpler process. Because converting differential circuit equations into their LaPlace Transform pairs is so labor saving (and by extension, error saving) it is well worth while to become familiar with the process.

The Definition

Learning to convert expressions to their LaPlace equivalent is straightforward. In every case we apply the definition of the LaPlace Transform:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

This expression says that the LaPlace Transform, $F(s)$, equals the integral of the time function, $f(t)$, times the transform function e^{-st}.

Ultimately the utility of the LaPlace Transform is to predict circuit behavior as a function of time, and by extension, using Bode's technique, to predict output amplitude and phase as a function of frequency. Further, the transform of the transfer function provides for plotting the poles and zeros of the transfer function, which in turn, lays the foundation for the Root Locus method of analyzing circuit stability as a function of amplitude and frequency. These topics will be covered in some detail as progression through the modules develops an ever increasing sophistication in the uses of the LaPlace Transform.

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The Basic Transform Pairs

Suppose we have a constant DC voltage of amplitude K. The task is to apply the definition and develop the LaPlace Transform of the constant K. Directly applying the definition:

$$f(t) = K \rightarrow F(s) = \int_0^\infty Ke^{-st}dt = K\int_0^\infty e^{-st}dt = -K\frac{e^{-st}}{s}\bigg|_0^\infty = \frac{K}{s}$$

Recall that $e^{-st} = \frac{1}{e^{st}}$ and therefore, as $t \rightarrow \infty$, $\frac{1}{e^{st}} \rightarrow 0$ and also that $e^0 = 1$.

$$K \leftrightarrow \frac{K}{s} \quad \textbf{(Transform Pair \#1)}$$
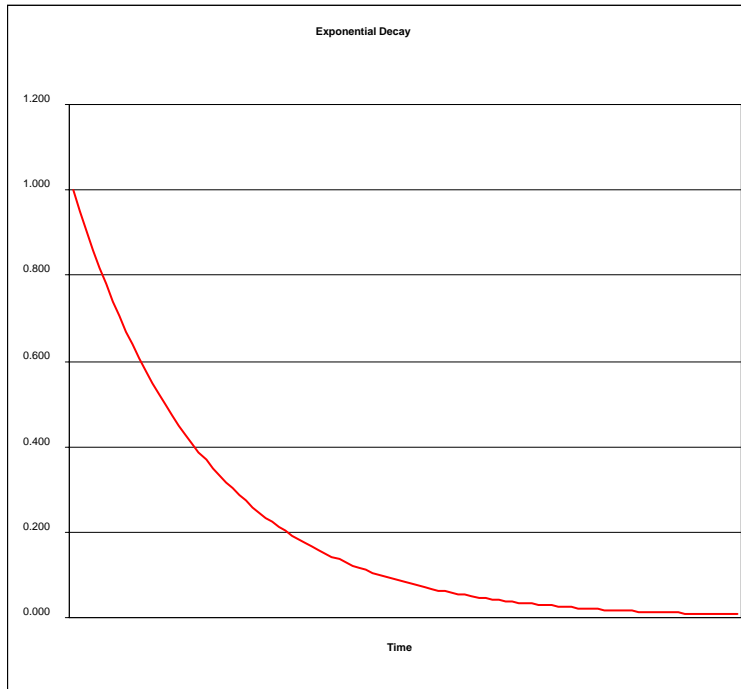
For example a constant voltage of 10vdc transforms to $10/s$; 413.32 transforms to $413.32/s$; $-22.87 \rightarrow -22.87/s$; etc.

Quite often in the physical world we are confronted with signals that exponentially decay over time can and be expressed as:

$$f(t) = e^{-\sigma t}$$

where σ represents some physical parameter and having the units of 1/sec or *rads* (radians). In electronics σ is usually a time-constant generated by either an RC (resistance/capacitance) or an RL (resistance/inductance) network. Recall that a single time-constant is defined when $\sigma t = 1$; the value of the time-constant being $1/\sigma$. When $\sigma t = 1$, $e^{-\sigma t} = .37$ meaning that the signal decays to 37% of its peak value in one time-constant.

Graphically, it looks like this:

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Exponential Decay

This particular form occurs so often in nature, especially when applied to signals that decay over time, that it is also assigned a transform.

Suppose there is an exponentially decaying voltage of value Ke^{-at}.

To find the transform we again directly apply the definition:

$$f(t) = Ke^{-at} \rightarrow F(s) = K\int_0^\infty e^{-at}e^{-st}dt = K\int_0^\infty e^{-(s+a)t}dt = -\frac{Ke^{-(s+a)t}}{(s+a)}\Big|_0^\infty = \frac{K}{(s+a)}$$

$$Ke^{-at} \leftrightarrow \frac{K}{s+a} \quad \textbf{(Transform pair \#2)}$$

For example:

$$10e^{-3t} \leftrightarrow \frac{10}{s+3}$$

$$-2.36e^{-6.78t} \leftrightarrow \frac{-2.36}{s+6.78}$$

and so forth.

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At this point we need to take a side excursion into Euler's Identities, as the use of these identities does two things: greatly simplifies the calculus of trigonometric functions by avoiding integration by parts and familiarizes us with the notational shorthand found in the literature.

All of the following four identities can be developed and established by using the processes of Maclaurin's infinite series, which is found in any beginner's Calculus text. Then by definition:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

Recall that the units on ω is in rads (radians), and that $\omega = 2\pi f$, also that $f = 1/t$ (units on f is in Hertz and t is in seconds), and as usual $j = \sqrt{-1}$. Also note that from either of the above identities:

$$e^{\pm j0} = \cos(0) \pm j\sin(0) = 1$$

Subtracting the lower from the upper, and rearranging, we get:

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

then adding the two, and rearranging, we get:

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Using the Identities

Suppose we have a simple sinusoid, such as:

$$f(t) = K\sin(\omega t)$$

then,

$$f(t) = K\frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

finding the LaPlace Transform by direct application then,

$$K\int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt = \frac{K}{2j}\left(\int_0^\infty e^{-(s-j\omega)t}dt - \int_0^\infty e^{-(s+j\omega)t}dt\right)$$

performing the integration, and evaluation at the limits:

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$$\frac{K}{2j}\left(-\frac{e^{-(s-j\omega)t}}{(s-j\omega)}\Big|_0^\infty+\frac{e^{-(s+j\omega)t}}{(s+j\omega)}\Big|_0^\infty\right)=\frac{K}{2j}\left(\frac{1}{(s-j\omega)}-\frac{1}{(s+j\omega)}\right)=\frac{K\omega}{s^2+\omega^2}$$

so

$$K\sin(\omega t)\leftrightarrow\frac{K\omega}{s^2+\omega^2}\quad\textbf{(Transform pair \#3)}$$

For example:

$$5.3\sin(7.72t)\leftrightarrow\frac{(5.3)(7.72)}{(s^2+7.72^2)}$$

$$6\sin(5t)\leftrightarrow\frac{30}{(s^2+25)}$$

$$-5.3\sin(7.72t)\leftrightarrow\frac{-40.92}{(s^2+59.6)}$$

Let $f(t)=K\cos(\omega t)$, then,

$$F(s)=K\int_0^\infty\cos(\omega t)e^{-st}dt=K\int_0^\infty\left(\frac{e^{j\omega t}+e^{-st}}{2}\right)e^{-st}dt=\frac{K}{2}\int_0^\infty e^{-(s-j\omega)t}dt+\frac{K}{2}\int_0^\infty e^{-(s+j\omega)t}dt$$

$$\frac{K}{2}\int_0^\infty e^{-(s-j\omega)t}dt+\frac{K}{2}\int_0^\infty e^{-(s+j\omega)t}=\frac{K}{2}\left(\frac{1}{(s-j\omega)}+\frac{1}{(s+j\omega)}\right)=\frac{Ks}{(s^2+\omega^2)}$$

Pair \#4 then,

$$K\cos(\omega t)\leftrightarrow\frac{Ks}{(s^2+\omega^2)}\quad\textbf{(Transform pair \#4)}$$

Examples:

$$4\cos(5t)\leftrightarrow\frac{4s}{(s^2+5^2)}=\frac{4s}{s^2+25}$$

$$-5.67\cos(4.3t)\leftrightarrow\frac{-5.67s}{s^2+18.49}$$

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Consider $f(t) = Ke^{-\sigma t}\sin(\omega t)$; applying the definition:

$$F(s) = K\int_0^\infty e^{-\sigma t}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt = \frac{K}{2j}\left(\int_0^\infty e^{-(s+\sigma-j\omega)t}dt - \int_0^\infty e^{-(s+\sigma+j\omega t)dt}\right)$$

$$\frac{K}{2j}\left(\int_0^\infty e^{-(s+\sigma-j\omega)t}dt - \int_0^\infty e^{-(s+\sigma+j\omega t)dt}\right) = \frac{K}{2j}\left(\frac{1}{(s+\sigma-j\omega)} - \frac{1}{(s+\sigma+j\omega)}\right)$$

$$\frac{K}{2j}\left(\frac{1}{(s+\sigma-j\omega)} - \frac{1}{(s+\sigma+j\omega)}\right) = \frac{K}{2j}\left(\frac{(s+\sigma+j\omega-s-\sigma+j\omega)}{((s+\sigma)-j\omega)((s+\sigma)+j\omega)}\right)$$

$$\frac{K}{2j}\left(\frac{(s+\sigma+j\omega-s-\sigma+j\omega)}{((s+\sigma)-j\omega)((s+\sigma)+j\omega)}\right) = \frac{K\omega}{(s+\sigma)^2+\omega^2}$$

So,

$$Ke^{-\sigma t}\sin(\omega t) \leftrightarrow \frac{K\omega}{(s+\sigma)^2+\omega^2} \quad \textbf{(Transform Pair \#5)}$$

Some examples:

$$5e^{-3t}\sin(6t) \leftrightarrow \frac{30}{(s+3)^2+36}$$

$$-6.54e^{-2.9t}\sin(5.54t) \leftrightarrow \frac{-36.23}{(s+2.9)^2+30.69}$$

As an aside, the interpretation of an electric signal written as:

$$-6.54e^{-2.9t}\sin(5.54t)$$

It has a peak value of ± 6.54 or $|6.54|$ (Why can peak value be written as $\pm$ in this case?)

It has a time constant of 340 milliseconds

Frequency in Hertz is about .88 (How does 5.54 get converted to .88?)

To be complete an electric signal will be written as:

$$Ke^{-\sigma t}\sin(\omega t + \theta)$$

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where θ is the phase angle of the signal when $t = 0$. Either or both σ and θ can be 0. A $\sigma = 0$ condition implies the signal does <u>not</u> decay with time. $\theta \neq 0$ means that $f(t)$ possesses some initial amplitude at $t = 0$.

Next is:

$$f(t) = Ke^{-\sigma t}\cos(\omega t)$$

Applying the definition:

$$F(s) = K\int_0^\infty e^{-\sigma t}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)e^{-st}dt = \frac{K}{2}\left(\int_0^\infty e^{-(s+\sigma-j\omega)t}dt + \int_0^\infty e^{-(s+\sigma+j\omega)t}dt\right)$$

Using the same procedures as previously, we get:

$$\frac{K}{2}\left(\frac{s+\sigma+j\omega+s+\sigma-j\omega}{(s+\sigma-j\omega)(s+\sigma+j\omega)}\right) = \frac{K(s+\sigma)}{\left((s+\sigma)^2+\omega^2\right)}$$
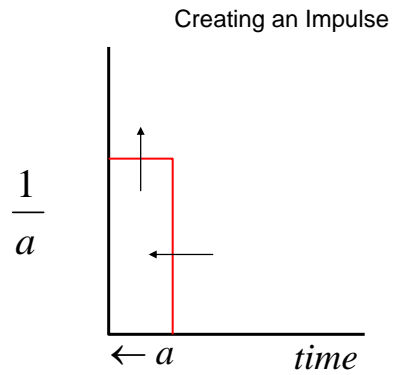
and therefore:

$$Ke^{-\sigma t}\cos(\omega t) \leftrightarrow \frac{K(s+\sigma)}{\left((s+\sigma)^2+\omega^2\right)} \quad \textbf{(Transform pair \# 6)}$$

Examples:

$$11.4e^{-3.5t}\cos(675t) \leftrightarrow \frac{11.4(s+3.5)}{(s+3.5)^2+455,625}$$

$$-6e^{-3.5t}\cos(5t) \leftrightarrow \frac{-6(s+3.5)}{(s+3.5)^2+5^2}$$

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Consider the function as shown below.

Creating an Impulse

It is clear that the area of the function is 1. At $t = 0$ however, the case is not so clear; therefore, let us define this function to be $\delta(t)$ and possessing the following properties:

$$\delta(t) = \left\{ \frac{\infty; t = 0}{0; elsewhere} \right\}$$

Since $\delta(t)$ only exists when its argument is 0, and recalling that the area under the curve is 1, finding the integral is reasoned as follows:

$$\int_0^{0^+} \delta(t) dt = 1$$

The transform then,

$$\int_0^{0^+} \delta(t) e^{-s0} dt = 1$$

so,

$$\delta(t) \leftrightarrow 1 \ \textbf{(Transform pair \#7)}$$

Examples:

$$1\delta(t) \leftrightarrow 1$$
$$5.73\delta(t) \leftrightarrow 5.73$$

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Saved by Reality

A word of caution is not out of order here. In actuality, $\delta(t)$ can never have a width of exactly 0; such a signal does not exist. $\delta(t)$ will always have some finite width, however narrow that may be (1 or 2 nanoseconds is pretty narrow). Therefore recognizing that the area of $\delta(t)$ is a constant for any width not exactly equal to 0 provides an extremely plausible argument.

Even though $\delta(t)$ has been defined to have an amplitude of ∞ at $t = 0$, in physical circuits ∞ is not realizable and some constraints must be applied to the idea of $\delta(t)$ to be useful practically. Circuit parameters, such as voltage, current and impedance, depend upon the variables of the circuit components, the power supply, and bandwidth of the signal and of the circuit. None of these variables can be infinite in a real circuit; therefore the practical or real limit on $\delta(t)$ is constrained to a far lesser value.

The most useful aspect of $\delta(t)$ is that its existence is limited to a specific very narrow temporal envelope, i.e., $0 \rightarrow 0^+$. In the limit, that envelope has no width and is $t = 0$, and that is an intellectual construct. For our purposes then, it follows that if $\delta(t)$ is applied to a signal $g(t)$ the result is some $h(0)$. As it will be clear later, the real utility here is that $\delta(t)$ becomes a tool to define the existence of signals at, and only at $t = 0$ (or as we will discuss in the next section, where the argument $(t - a) = 0$). The theoretical and mathematical subtleties of this function will not be utilized beyond its utility as a place holder in time.

Time Shifting, the Unit Step and the Rectangular Pulse
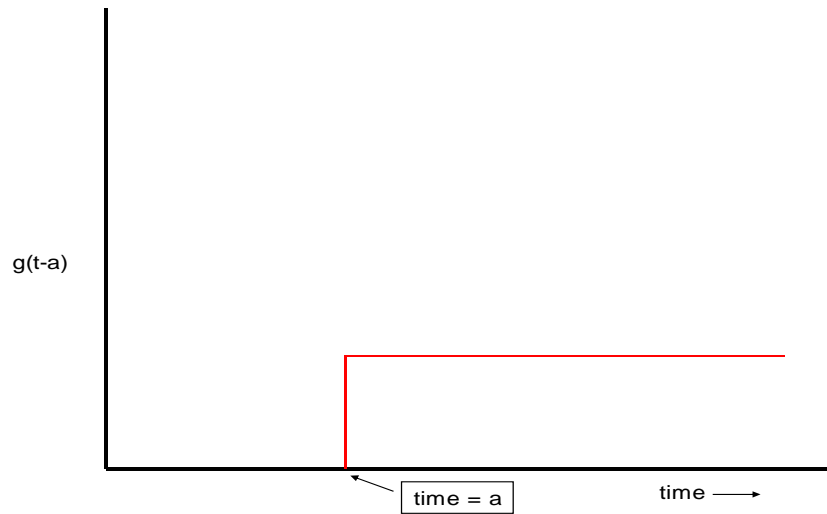
Before we can mathematically define such a model for our purposes here, it is necessary to introduce the modeling of time shifting. For example; suppose we wish to define a signal to begin when the switch was thrown and the circuit is energized. Let such a signal be known as g, and let us agree that g possesses an argument (t), and further $g(t)$ does not exist until the value of the argument $(t) = 0$. Or,

$$g(t) = 0 \text{ for all } t < 0$$
$$g(t) = 1 \text{ for all } t \geq 0$$

Further, let us define an argument on g, such that when we write $g(t - a)$ we agree that the signal cannot appear until the value $t - a = 0$. Such that

$$g(t - a) = 0 \text{ for all } (t - a) < 0$$
$$g(t - a) = 1 \text{ for all } (t - a) \geq 0$$

For example:

g(t-a)

time = a

time ⟶

Notice that the signal we just agreed to define has an amplitude of either 0 or 1. When $(t-a) \geq 0$ we will call this signal with an amplitude of 1 as a unit step.

In the literature the unit step is frequently seen as $u(t)$ or as $u(t-a)$ when time shifted. We will now follow that convention.

Using the definition, the LaPlace transform of the unit step is:

$$F(s) = \int_0^\infty u(t)e^{-st} dt$$

Since $u(t) = 1$ for $t \geq 0$ and 0 everywhere else, we can re-write the above as:

$$F(s) = \int_0^\infty e^{-st} dt = \frac{1}{s}$$

$$u(t) \leftrightarrow \frac{1}{s}$$

by extension:

$$Ku(t) \leftrightarrow \frac{K}{s}$$

Consider a time shifted step, $u(t-a)$. Since the signal does not exist until $t = a$, the limits of integration are changed to reflect this reality; therefore let $f(t) = u(t-a)$ and apply the definition:

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$$F(s) = \int_a^\infty u(t-a)e^{-st}\,dt = \left.\frac{-e^{-st}}{s}\right|_a^\infty = 0 + \frac{e^{-as}}{s}$$

So,

$$u(t-a) \leftrightarrow \frac{e^{-as}}{s}$$

and by extension:

$$Ku(t-a) \leftrightarrow \frac{Ke^{-as}}{s}$$

Examples:

$$4u(t-3) \leftrightarrow \frac{4e^{-3s}}{s}$$

$$-9.033u(t-5.76) \leftrightarrow \frac{-9.033e^{-5.76s}}{s}$$

Pay attention to the exponent $-as$; it is a <u>codeword</u>, and the codeword means the signal does not start until $t = a$. That is all it means; a is the time offset. Notice that the exponent $-as$ has a different interpretation from the exponent $-st$; one is a specific time and the other is general. Time offset, $u(t-a)$, can be applied to any signal in the chart of transform pairs in Table 1 with a corresponding right hand side factor of e^{-sa}
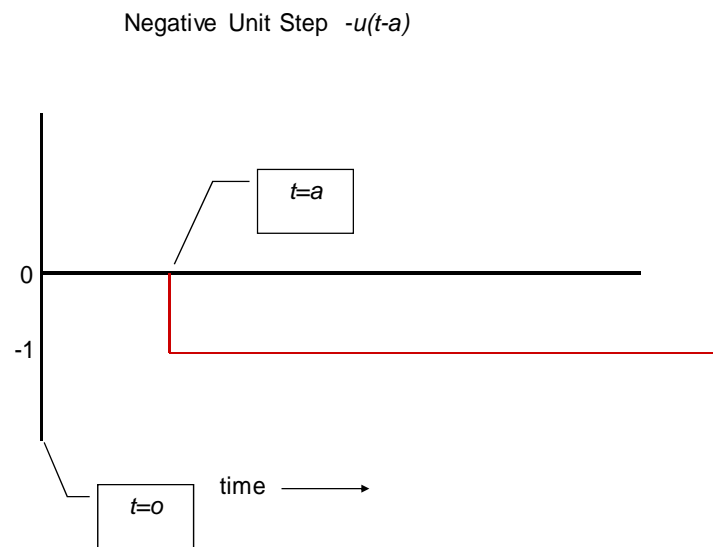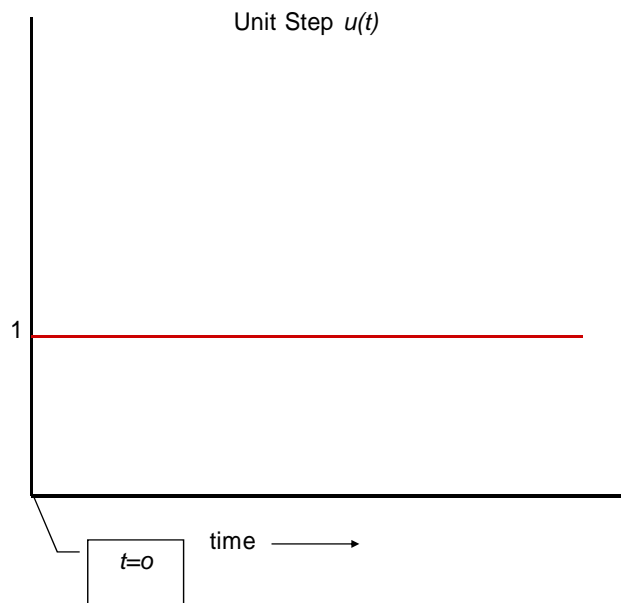
An example is:

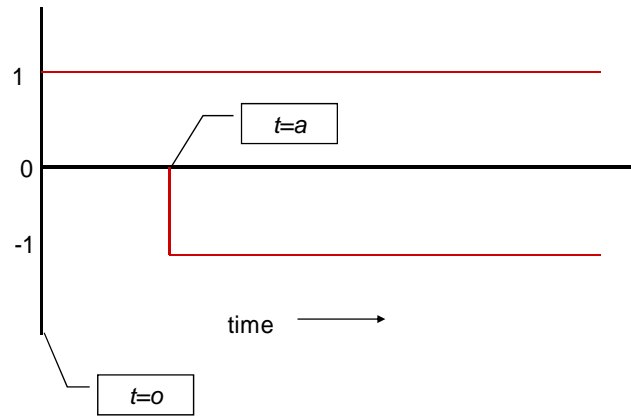$$Ke^{-\sigma t}\sin(\omega t)u(t-a) \leftrightarrow \frac{K\omega e^{-sa}}{(s+\sigma)^2 + \omega^2}$$

or,

$$11e^{-5t}\sin(7t)u(t-10) \leftrightarrow \frac{77e^{-10s}}{(s+5)^2 + 49}$$

The ability to mathematically model time shifting gets one step closer to modeling a rectangular pulse. The next step is to add two unit steps, one positive and the other negative.
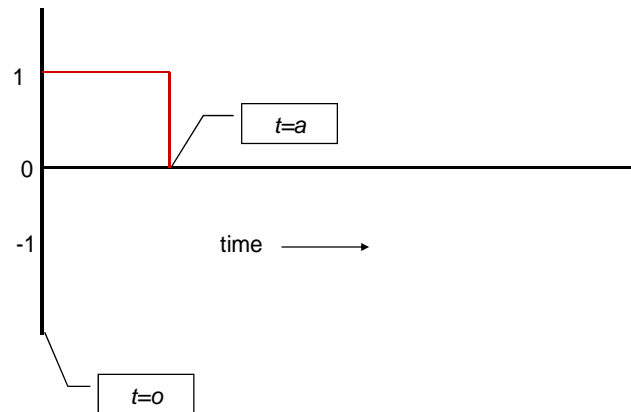
Conceptually, this signal can be constructed this way:

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Unit Step *u(t)*

1

t=o

time ⟶

Negative Unit Step *-u(t-a)*

t=a

0

-1

t=o

time ⟶

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Coincident $u(t)$ & $-u(t-a)$

Algebraic Result of $u(t)$ & $-u(t-a)$

The algebraic addition of a unit step and a negative time shifted unit step creates the foundation of the signal we are looking for. Mathematically it is modeled by:

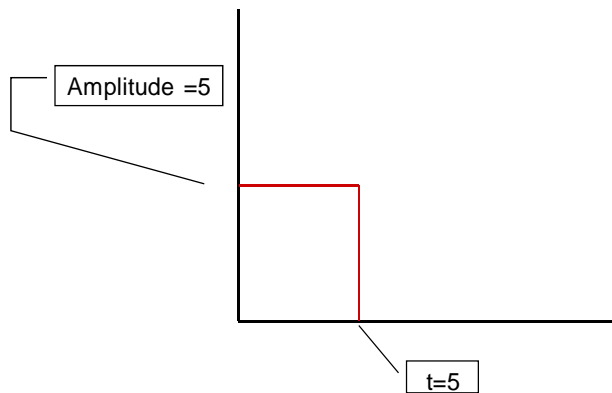$$u(t) - u(t-a) \leftrightarrow \frac{1}{s} - \frac{e^{-as}}{s}$$

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Again, by extension:

$$K(u(t) - u(t - a)) \leftrightarrow \frac{K}{s}\left(1 - e^{-as}\right)$$

For example:

$$5(u(t) - u(t - 5))$$

Looks like:

with a transform of :

$$\frac{5}{s}\left(1 - e^{-5s}\right)$$

The Transform of the Derivative and the Integral of $f(t)$

Suppose we wish to find the LaPlace transform of the derivative of a function, $f'(t)$. Again we will begin with the definition:

$$F(s) = \int_0^\infty f'(t)e^{-st}dt$$

Integrating by parts and allowing the variables to be distributed thus:

$$u = e^{-st} \qquad v = \int_0^\infty f'(t)dt$$
$$du = -se^{-st} \qquad dv = f'(t)$$

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$$F(s) = \int_0^\infty f'(t)e^{-st}\,dt = e^{-st}\int_0^\infty f'(t)\,dt + s\int_0^\infty e^{-st}\int_0^\infty f'(t)\,dt$$

since,

$$\int_0^\infty f'(t)\,dt = f(t)$$

re-writing becomes:

$$F(s) = \int_0^\infty f'(t)e^{-st}\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

and since by definition:

$$\int_0^\infty f(t)e^{-st}\,dt = F(s)$$

then,

$$\int_0^\infty f'(t)e^{-st}\,dt = -f(0) + sF(s)$$

or,

$$f'(t) \leftrightarrow sF(s) - f(0) \quad \textbf{(Transform pair \#9)}$$

Some examples:

$$C\frac{dv_c}{dt} \leftrightarrow sCv_c(s) - v_c(0)$$

$$L\frac{di}{dt} \leftrightarrow sLi(s) - i(0)$$

As a side bar, since $v = iz$, it follows that sL and $\dfrac{1}{sC}$ are impedances.

The next transform that we will consider in this part is the integral of $f(t)$.

Applying the definition, as usual:

$$F(s) = \int_0^\infty \left(\int f(t) \right) e^{-st}\,dt$$

Again, we will integrate by parts, and choose:

$$u = \int f(t)\,dt \quad v = \frac{-e^{-st}}{s}$$
$$du = f(t) \quad dv = e^{-st}$$

then,

$$\int_0^\infty \left(\int f(t) \right) e^{-st}\,dt = \int f(t)\,dt\left(\frac{-e^{-st}}{s}\Big|_0^\infty \right) + \frac{1}{s}\int f(t)e^{-st}\,dt$$

Evaluating the first term on the RHS:

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$$\left(\int_0^\infty f(t)dt \right)(0) - \left(\int_0^0 f(0)dt \right)(1) = 0$$

the transform becomes:

$$\int_0^\infty \left(\int f(t) \right) e^{-st} dt = 0 + \frac{F(s)}{s}$$

or,

$$\int f(t)dt \leftrightarrow \frac{F(s)}{s} \quad \textbf{(Transform pair \#10)}$$

a circuit example is:

$$\frac{1}{L} \int v_L(t)dt \leftrightarrow \frac{v_L(s)}{sL}$$

For a simple RLC series circuit, the differential equation satisfying Kirchhoff's Voltage Law is:

$$iR + \frac{1}{C} \int idt + L\frac{di}{dt} = v_{in}$$

By direct application of the transform pairs, the above converts to (after re-arranging):

$$\left(\frac{L}{s} \right)\left(s^2 + \frac{R}{L}s + \frac{1}{LC} \right)i(s) = v_{in}(s)$$

As we shall see, this quadratic form plays a major role in circuit analysis and design. It becomes the focus of our attention particularly when issues of circuit stability occur.

Another transform for Part 1, which is really a property of the mathematics, is the transform of $af(t) + bg(t)$. Applying the definition:

$$\int_0^\infty \left(af(t) + bg(t) \right)e^{-st} dt = a\int_0^\infty f(t)e^{-st} dt + b\int_0^\infty g(t)e^{-st} dt = aF(s) + bG(s)$$

$$af(t) + bg(t) \leftrightarrow aF(s) + bG(s) \quad \textbf{(Transform Pair \#11)}$$

Examples:

$$3 + 5\cos(200t) \leftrightarrow \frac{3}{s} + \frac{5s}{s^2 + 200^2}$$

$$3 + 5\cos(200t) \leftrightarrow \frac{8\left(s^2 + 15000\right)}{s\left(s^2 + 200^2\right)}$$

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In subsequent modules we will see how denominator factoring prevents the masking of the transformation pair identity and partial fraction expansion reduces the right side of the lower of the above two transforms to the right side of the upper.

In Part 2 of this series, we will begin to use these transforms for constructing circuit equations and simple transfer functions. Also any other transforms we might need for analysis will be developed as necessary. The emphasis will be on learning by examples. Of course theoretical foundations will be provided when needed.

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Summarizing the pairs for Part 1

Transform	$f(t)$	$F(s)$
1	K	$\dfrac{K}{s}$
2	$Ke^{-\sigma t}$	$\dfrac{K}{s+\sigma}$
3	$K\sin(\omega t)$	$\dfrac{K\omega}{s^2+\omega^2}$
4	$K\cos(\omega t)$	$\dfrac{Ks}{s^2+\omega^2}$
5	$Ke^{-\sigma t}\sin(\omega t)$	$\dfrac{K\omega}{(s+\sigma)^2+\omega^2}$
6	$Ke^{-\sigma t}\cos(\omega t)$	$\dfrac{K(s+\sigma)}{\left((s+\sigma)^2+\omega^2\right)}$
7	$\delta(t)$	1
7a*	$K\delta(t)$	K
8	$Ku(t-a)$	$\dfrac{Ke^{-as}}{s}$
9	$f'(t)$	$sF(s)-f(0)$
10	$\int f(t)dt$	$\dfrac{F(s)}{s}$
11	$af(t)+bg(t)$	$aF(s)+bG(s)$

(*) K is preserved for practical circuit reasons, not for theoretical reasons as $K*\infty$ is approximately equal to ∞

Table 1

Table 1 is not all inclusive and other pairs will be examined and added when needed. But for beginning analysis purposes Table 1 is adequate.

It is very important to understand that to be able transform any $F(s)$ to an $f(t)$, $F(s)$ must be reduced to one of the forms so far developed. If it is not in one of these forms it cannot be operated on until it is. Study the right hand side functions, they identify the left hand side functions.

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