LaPlace Transforms in Design and Analysis of Circuits

Part 4: Frequency and Phase Analysis

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Credit: 4 PDH

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LaPlace Transforms in Design and Analysis of Circuits©
Part 4

by Tom Bertenshaw

Frequency and Phase Analysis

Domain of "s"

Suppose there is a quadratic in a transfer function denominator, such as:

$$s^2 + 6s + 13$$

The roots of that expression are:

$$s = -3 \pm j2$$

For reasons of stability, which will be explored in detail in subsequent modules, we will restrict the signs in the quadratic to be positive, i.e. the roots lie in quadrants II or III, meaning to the left of the $j\omega$ axis or on the $j\omega$ axis), i.e., $0 \pm j\omega$). That is a general, and prudent design constraint. A little thought about the behavior of $f(t)$ as a function of $e^{\sigma t}$ provides the reason.

No matter how many iterations of finding the roots of a large quantity of quadratic equations where the roots are complex, we will consistently come to the conclusion that the domain of s is the entire complex plane, and specifically for our purposes $s = -\sigma \pm j\omega$ is the general solution of a complex quadratic with roots to the left of the $j\omega$ axis.

σ is a real number that is a function of the time constant, and ω is the frequency of oscillation in rads/s (when $\sigma \neq 0$ (or equal to 3 as in the example above), ω is the damped frequency (or equal to 2 as in the example above); which is less than resonance: $\sqrt{13}$). The values of both σ & ω arise from the values of the circuit components.

Extending this argument, consider the core variable in any LaPlace transform:

$$e^{-st} = e^{-(\sigma \mp j\omega)t} = e^{-\sigma t} e^{\mp j\omega t}$$

Since exponents are unitless, the units on both σ & ω must be t^{-1}, and indeed they are. Question: can you show this is true? Ponder: Do you see a shadow of relationship between a LaPlace transform and a Fourier transform from $e^{-\sigma t} e^{\mp j\omega t}$?

The point of all this is that since the LaPlace transform can be written as a function of $e^{-\sigma t} e^{-j\omega t}$, we can legitimately develop a method for expressing the response of the transfer function as a function of frequency, and by extension, its phase at any given frequency.

So far we have examined the case where the roots are complex. For the real roots, usually the denominator of those expressions will be of the first order of the form:

$$s+\sigma$$

Repeated real roots are an exception and they are of an order:

$$(s+\sigma)^m \quad m = 2,3,.........n$$

Repeated roots **do not** present the same degree of difficulty in frequency analysis as their inversion does when finding partial fraction expansions (PFE) for time domain analysis. Some excellent engineers that I have been associated with over the years will argue that exact repeated roots are not possible, so in the practical case they never have to be dealt with. The reasoning behind that is that no two time constants can ever contain components whose values are exactly the same to n decimal places, i.e., there will always be a slight amount of ambiguity in value, and the analyst can take advantage of that to use only non-repeated roots regardless of how close any n roots are to each other. However, FAPP (For All Practical Purposes[1]) using n roots that are a virtual small value apart make **no discernable** difference in the output from n repeated identical roots.

How to treat repeated roots when inverting into the time domain is another of those cases where you have to be aware that the method you choose may lead to ridiculous amplitudes. For the sake of prudence, when we invert in this series of modules we will stick to identical roots and use the formal method of differentiation for PFE. Choosing methods is not a consideration when dealing with frequency analysis as there is no need for PFE.

In general, a transfer function will be a combination of first order, second order and any repeated root factors in both the numerator and the denominator:

$$\frac{(s+z_1)(s+z_2)..........(s+z_m)(s+z_r)^p\left((s+\sigma_1)^2+\omega_1^2\right).......\left((s+\sigma_m)^2+\omega_m^2\right)}{(s+p_1)(s+p_2)..........(s+p_n)(s+p_r)^k\left((s+\alpha_1)^2+\omega_{d1}^2\right)......\left((s+\alpha_n)^2+\omega_{dn}^2\right)} \quad \leftarrow \text{Eq. 1}$$

Where $s=-z_x$ are the zeros of the function (causes the function to be zero), and $s=-p_x$ are the poles (like a telephone pole andcauses the function to spike). It should be clear that if any roots are complex, a quadratic will appear, and all others will be first order roots or repeated roots that are real (repeated complex roots are also possible but we will leave that case to the future).

[1] John S. Bell, 1928-1990, Physicist extraordinaire

Obviously a frequency is a constituent of the complex case, but what about the other cases? It can be shown that the units of a time constant (RC or L/R) is time. It follows that the reciprocal of the time constant is frequency. The frequency associated with the reciprocal of the time constant is called the "break frequency" for reasons that will be apparent shortly.

The Transfer Function as a Logarithm

For the purposes of analysis or design, if we consider the behavior of a transfer function as a function of ω (or alternately, the reciprocal of RC or L/R) we will find that an accurate picture of amplitude versus frequency and phase versus frequency emerges.

First a digression -- Recall that a decibel is defined as (P_{xx} is power in watts)”

$$10\log\frac{P_{out}}{P_{in}}$$

which can be re-written as:

$$10\log\frac{P_{out}}{P_{in}} = 10\log\frac{V_{out}^2}{V_{in}^2} = 20\log\frac{V_{out}}{V_{in}}$$

The above definition assumes that the input and output resistances are approximately equal $\left(\frac{R_{in}}{R_{out}} \approx 1\right)$, allowing the substitution of $\frac{V_{xx}^2}{R_{xx}}$ for P_{xx}. That is not an overly confining assumption in a passive network as it allows for maximum power transfer; a generally desirable design feature.

Any transfer function in the LaPlace domain is the ratio of:

$$\frac{N(s)}{D(s)}$$

(see Eq. 1) and that ratio can certainly be constructed to express the relationship of V_{out} to V_{in}. We can re-write the transfer function as:

$$20\log\frac{V_{out}}{V_{in}} = 20\log N(s) - 20\log D(s)$$

The Details

Assume a transfer function of the form of:

$$\frac{K}{s+\sigma}$$

Re-write that to obtain the form:

$$\frac{K/\sigma}{\left(1+\frac{s}{\sigma}\right)}$$

Since the units of $\sigma = \frac{1}{t}$, let σ equal ω_o. Then we substitute $j\omega$ for s, and the transfer function now looks like:

$$\frac{\frac{K}{\omega_0}}{\left(1+\frac{j\omega}{\omega_o}\right)}$$

Bear in mind that $\sigma = \omega_o = \frac{1}{RC}$ or $\frac{L}{R}$, so as a designer you always retain control over that value.

$\left(1+\frac{j\omega}{\omega_o}\right)$ is complex, so we will convert that to polar notation:

$$\left(1+\frac{j\omega}{\omega_o}\right) = \sqrt{1+\frac{\omega^2}{\omega_o^2}}\ \angle\tan^{-1}\left(\frac{\omega}{\omega_o}\right)$$

Taking the logarithm of both sides, the magnitude of the transfer function is:

$$20\log\frac{V_{out}}{V_{in}} = 20\log K - 20\log\omega_o - 20\log\sqrt{1+\frac{\omega^2}{\omega_o^2}}$$

The phase angle as a function of frequency is $-\arctan\left(\frac{\omega}{\omega_o}\right)$ (why *minus* arctan? Because the expression is in the denominator and the phases of the constants are both zero.) Phase angle is always taken to mean the phase of the output with respect to the input.

Lo-Pass Filter

As an example, consider:

$$\frac{10}{s+10}$$

For purposes of illustration, the left hand side of transfer functions is omitted but understood to be $\frac{V_{out}}{V_{in}}$ for our present purposes. Later on the left hand, parameters may change and if/when they do the change will be identified.

Re-arranging:

$$\frac{1}{1+\frac{s}{10}} \rightarrow \frac{1}{1+\frac{j\omega}{10}} \rightarrow \frac{1}{\sqrt{1+\frac{\omega^2}{100}}\angle \tan^{-1}\left(\frac{\omega}{10}\right)}$$

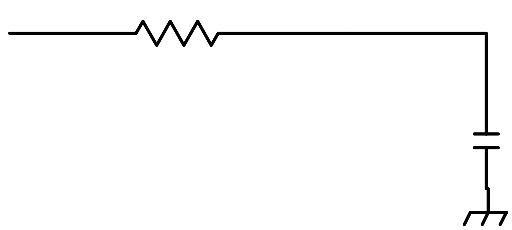
Converting to decibels:

$$Magnitude(db) = 20\log\left(\frac{1}{\sqrt{1+\frac{\omega^2}{100}}}\right) \rightarrow 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{100}}\right)$$

$$phase(\omega) = \arctan(0) - \arctan\left(\frac{\omega}{10}\right)$$

A single pole lo-pass filter passes frequencies below the break frequency relatively undiminished in magnitude and with relatively minor phase change.

It may be helpful to connect the transfer function to a circuit to help visualize what is going on here.



In the above circuit assume V_{out} is taken across the capacitor and that $RC = .1$. In that case then:

$$\frac{V_{out}}{V_{in}} = \frac{10}{s+10}$$

And by the process developed above:

$$Magnitude(db) = 20\log\left(\frac{1}{\sqrt{1+\frac{\omega^2}{100}}}\right) \rightarrow 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{100}}\right)$$

$$phase(\omega) = \arctan(0) - \arctan\left(\frac{\omega}{10}\right)$$

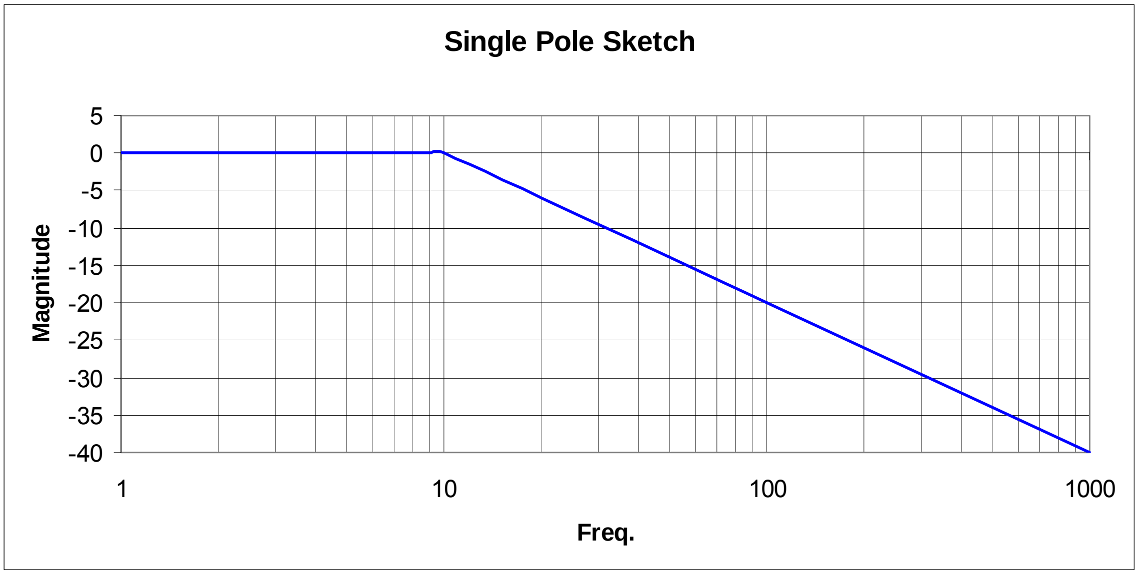
Recalling that $X_C = \frac{-j}{\omega C}$, the transfer function as a function of ω from inspecting the circuit schematic is:

$$\frac{V_{out}}{V_{in}} = \frac{-j}{\omega C\left(R - \frac{j}{\omega C}\right)} = \frac{-j}{(\omega RC - j)} \quad \leftarrow \text{circuit equation}$$

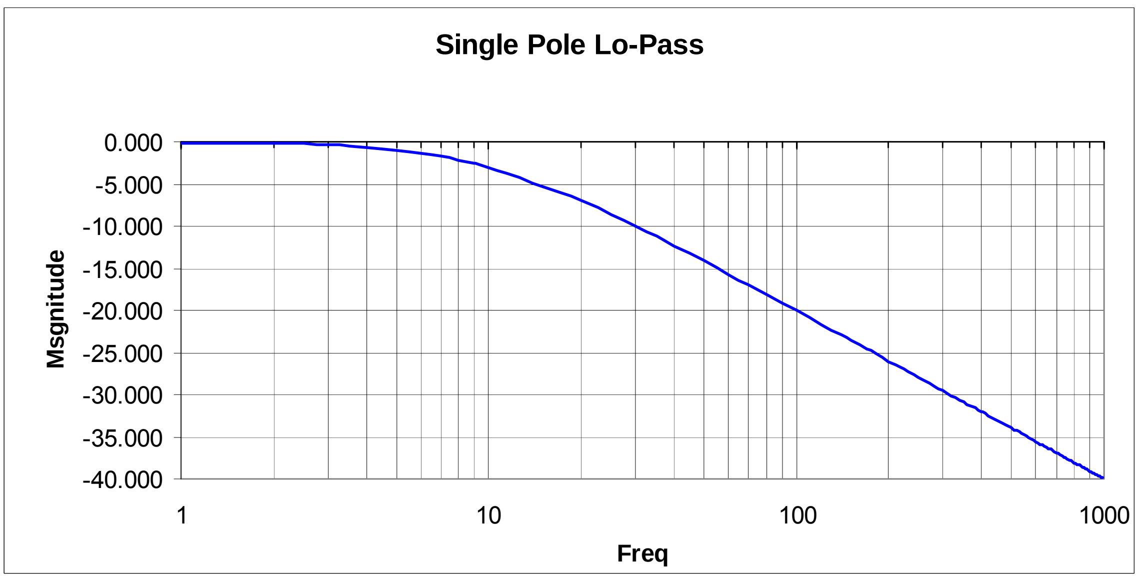
Then as a function of ω, both the magnitude in db and the phase start out as zero when $\omega = 0$, as verified by inspection of both the circuit equation and the magnitude and phase equations. The phase ends at -90^o as $\omega \to \infty$, and the magnitude continues to decline by -20db per decade, again as verified by inspection of both equations.

Briefly sketching the magnitude: when $\omega << 10$, the magnitude is $0db$ FAPP[2]; at $\omega = 10$, the magnitude is $-20\log\sqrt{2} = -3db$; when $\omega >> 10$, the magnitude for all practical purposes is $-20\log\left(\frac{\omega}{10}\right)$ (with a $-20db$ .per decade slope, or "roll-off"). The "break frequency" is the frequency at which $\omega = \omega_o$, i.e., the $-3db$ point.

[2] For All Practical Purposes

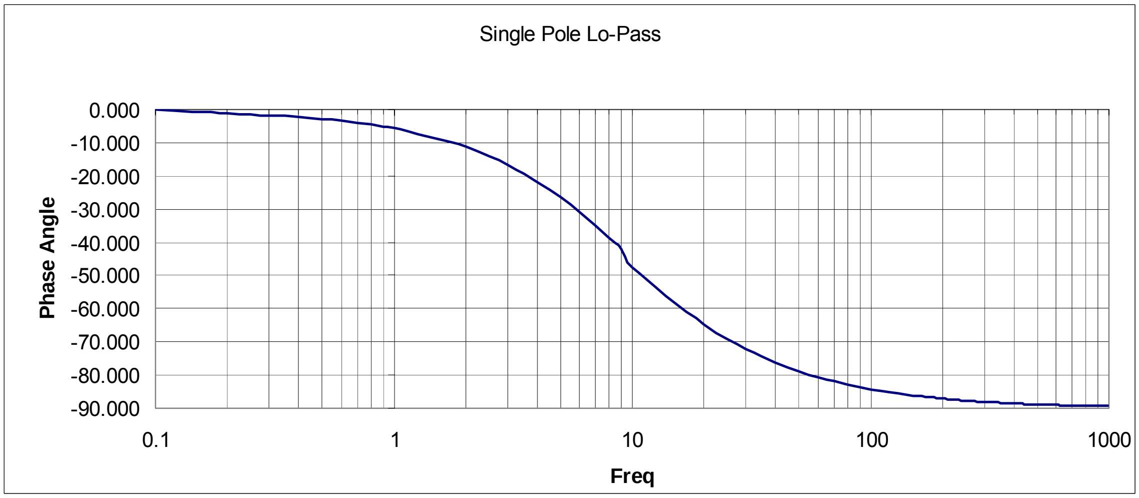


As plotted from the equation:



As can be seen from the above Bode[3] (Bo-dee) plot, the output magnitude begins to roll off as ω approaches the break frequency (in this case 10 rads/s). At the break frequency the magnitude is at -3db and rolls off at -20db per decade. Please note that this technique plots the output magnitude versus frequency, but does not address the frequency content of the output. That aspect of system response is left to Fourier analysis which is the subject of a different set of modules.

[3] Named for H. W. Bode, 1905-1982, who developed the technique.

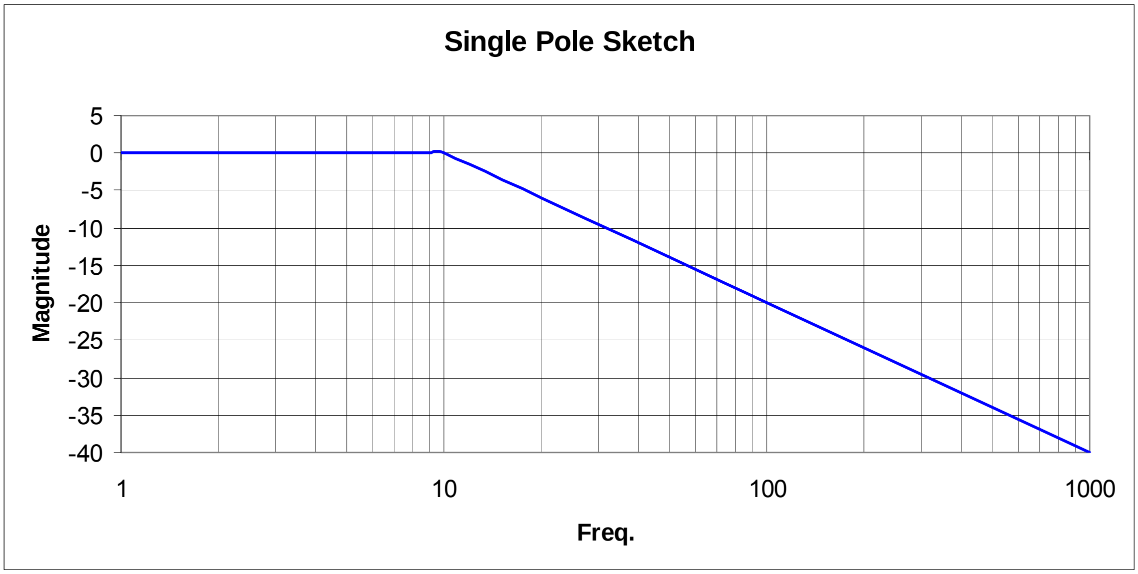


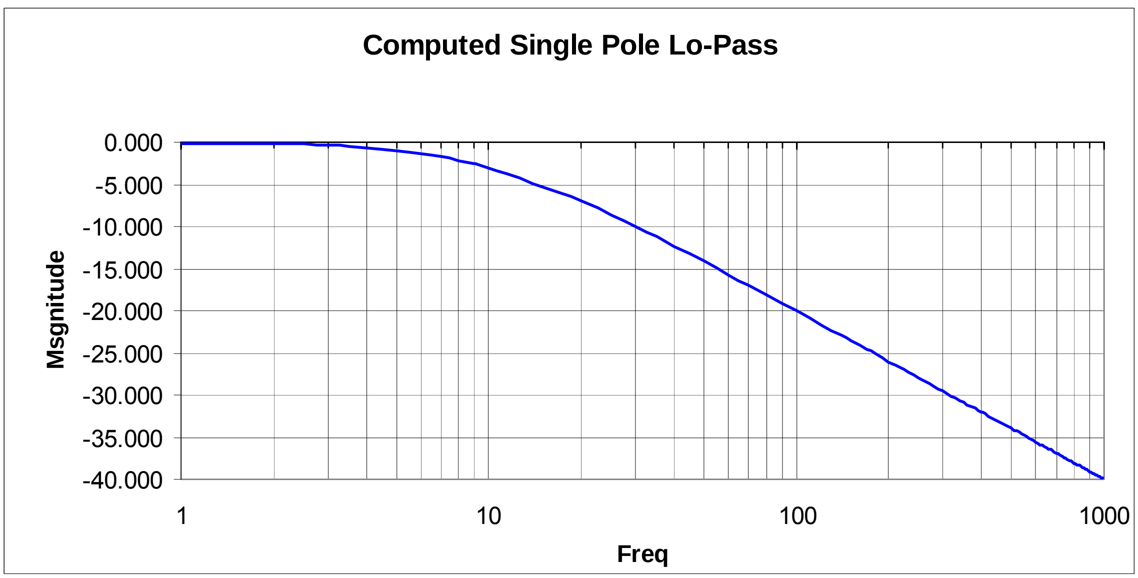
As expected, the phase of the output with respect to the input varies from 0 to -90^o as a function of ω. At very low frequencies the capacitor acts like an open and the input is the output. At high frequencies the capacitor begins to act like a short, and by the voltage division rule an ever larger percentage of the input is dropped across the resistor, while the voltage that is dropped across the capacitor approaches the -90^o rail.

Both the magnitude and the phase plots are readily hand sketched for a rapid peek at the performance envelope. For the magnitude sketch use the following "rule of thumb":

a. Begin at the lowest frequency of interest, at the magnitude of that frequency.
b. Draw a line horizontally to the first break frequency. If the lowest frequency is also a break frequency, draw a line at a slope of ∓ 20 db/decade per pole or zero (- for a pole, + for a zero).
c. From the break frequency to the next (or to the terminal frequency if there are no further break frequencies), draw -20db/decade per pole sloped lined or a+20db/decade per zero sloped line $\left(\pm 20N \log \sqrt{1 + \left(\frac{\omega}{\omega_o} \right)^2} \right)$ where N is the number of zeros or poles at that frequency. Remember that the addition of -20Ndb/decade with a +20Ndb/decade equals a horizontal line.
d. Repeat step c until all break frequencies are accounted for.

Examine both the sketched and the computed plot below. The greatest error occurs at the break frequency of 10 rads/s; the computed magnitude is -3db from the magnitude sketch. That is easy to remember; your error is max at the break frequency and it is $\mp 3db$ per pole or zero at that frequency.





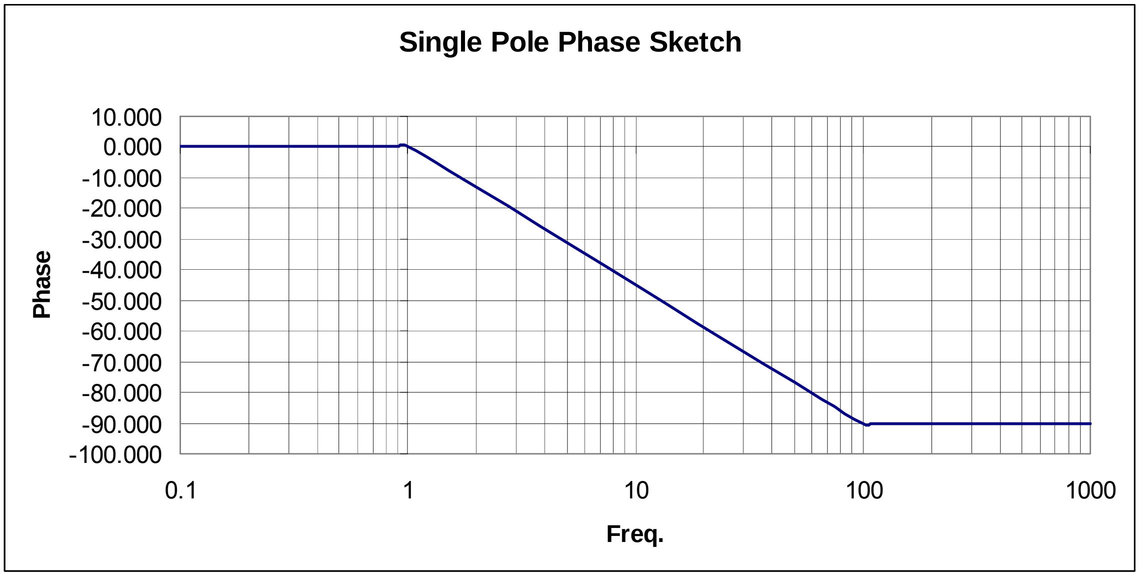
Rule of thumb for sketching the phase plot is:

a. Since $\tan^{-1}\dfrac{\omega}{\omega_o} = 1$ when $\omega = \omega_o$ assign +N45^o to a zero break frequency and a -N45^o to a pole break frequency; N being the number of poles or zeros at that break frequency.

b. One decade back from any break frequency, assign the phase to 0 for a pole or zero.

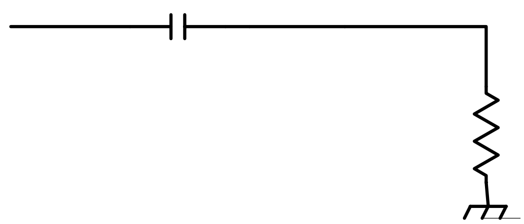
c. One decade above the break frequency, assign the phase to be +N90^o for a zero or -N90^o for a pole.

d. Connect the dots. At points of ambiguity (for example a frequency that is one decade above a pole ω_o while simultaneously being one decade back from a zero ω_o), it is best to compute the value. Phase plots can be very tricky to sketch, so calculation where ambiguity exists is recommended. But it is clear that in the above stated case, as $\omega \to \infty$ the sum will be zero degrees.



Sketching the magnitude and phase plots was particularly useful and time conserving back in the slide rule days. However, using graphing calculators and laptops performing a calculated plot is almost as fast, while being far more accurate. The graphs in this module were made using MS Excel®. It remains a matter of personal choice whether you prefer a sketch or a calculated plot. Calculated plots are required where accuracy is an issue.

The simple counterpart to the single pole lo-pass is the single pole hi-pass.



The general transfer function for this filter is:

$$\frac{R}{\left(R+\frac{1}{sC}\right)}=\frac{s}{\left(s+\frac{1}{RC}\right)}$$

Re-arranging into a format suitable for Bode analysis by substituting $j\omega$ for s and ω_o for $\dfrac{1}{RC}$:

$$\frac{j\omega}{(\omega_o + j\omega)} = \frac{j\dfrac{\omega}{\omega_o}}{\left(1 + j\dfrac{\omega}{\omega_o}\right)} = \frac{\dfrac{\omega}{\omega_o}\angle 90^o}{\sqrt{1+\dfrac{\omega^2}{\omega_o^2}}\angle \tan^{-1}\dfrac{\omega}{\omega_o}}$$

Converting to decibel format:

$$20\log\frac{Output}{Input} = 20\log\frac{\omega}{\omega_o} - 20\log\left(\sqrt{1+\left(\frac{\omega}{\omega_o}\right)^2}\right)$$

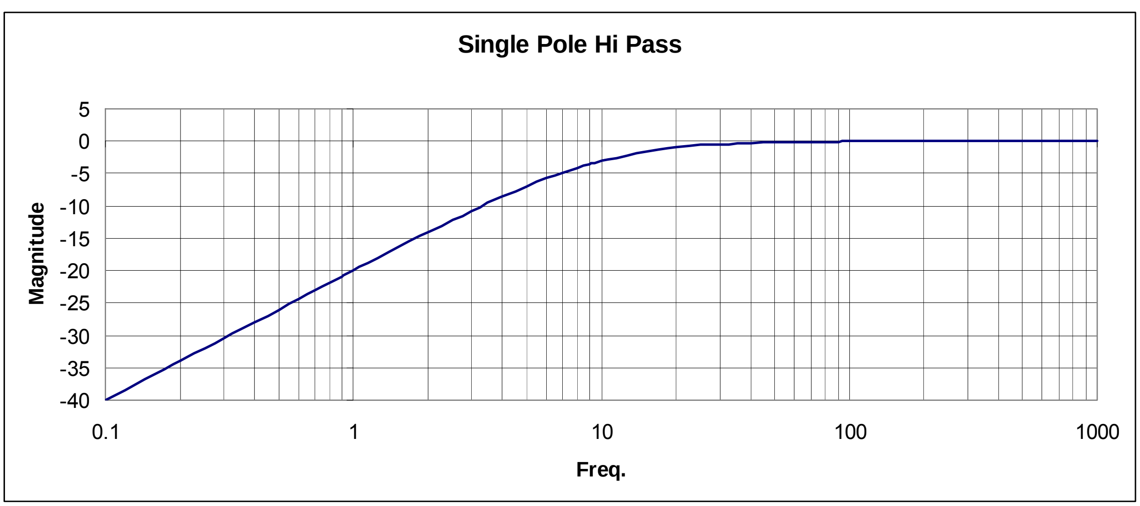
$$Phase = 90^o - \tan^{-1}\left(\frac{\omega}{\omega_o}\right)$$

Using the same break frequency as the single pole lo-pass:

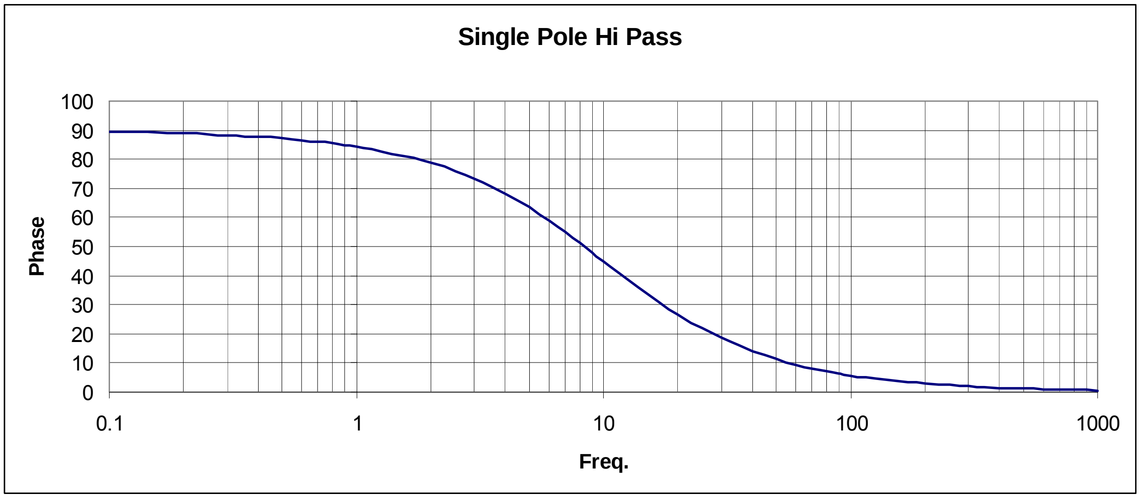
$$Magnitude = 20\log\left(\frac{\omega}{10}\right) - 20\log\sqrt{1+\frac{\omega^2}{100}}$$

$$Phase = 90 - \arctan\left(\frac{\omega}{10}\right)$$

Plotting these:



The roll-on is an expected +20db/decade and the characteristic -3b magnitude at the break frequency is evident. While this circuit illustrates the basic performance of this class of filters, like the single pole lo-pass, the magnitude change per decade is lack luster. In both cases repeated poles at the break frequency are needed to attain useful performance.



The phase shift pattern mirrors the single pole lo-pass in profile, but the limits are reversed as indicated by both the plotting equation and the schematic. We should expect the drop across the resistor to be 90^o out of phase at frequencies well below break, and to asymptotically approach zero as $\omega \to \infty$.

Multiple Poles
Lo-Pass

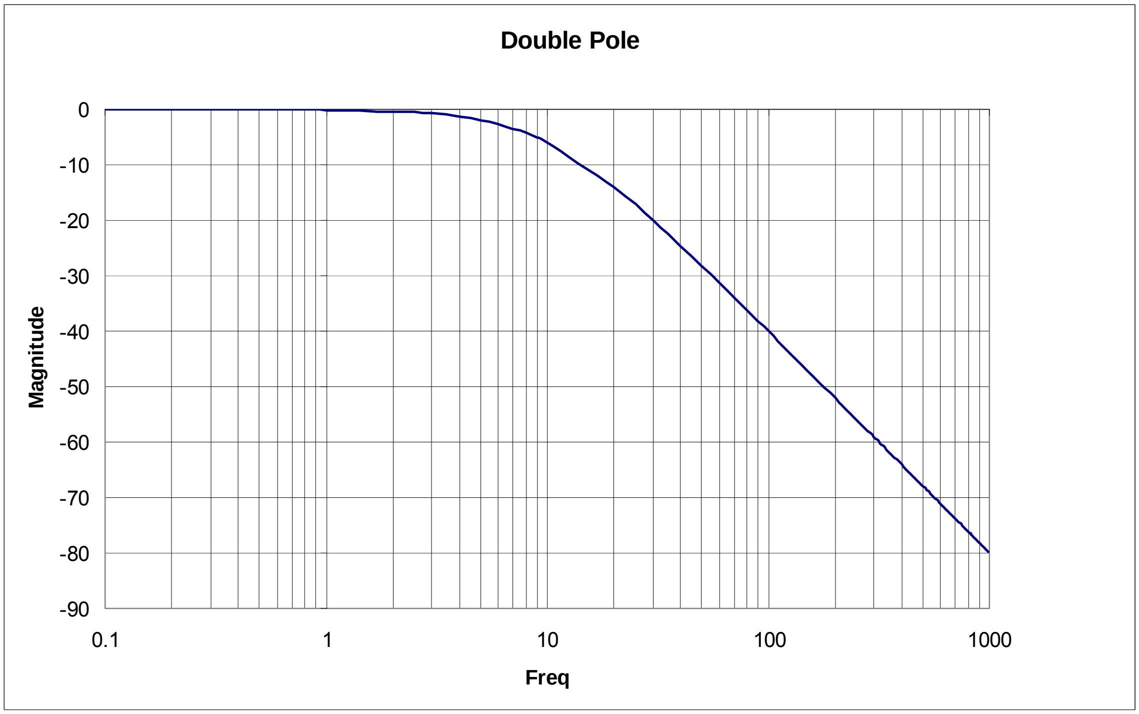
Let us consider the double pole transfer function with a break frequency of 10 rads/s. (Remember, for all practical purposes the poles need not be exactly superimposed, merely close enough so that treating them as exact has no appreciable effect on the outcome prediction).

$$\frac{Output}{Input} = \frac{100}{(s+10)^2}$$

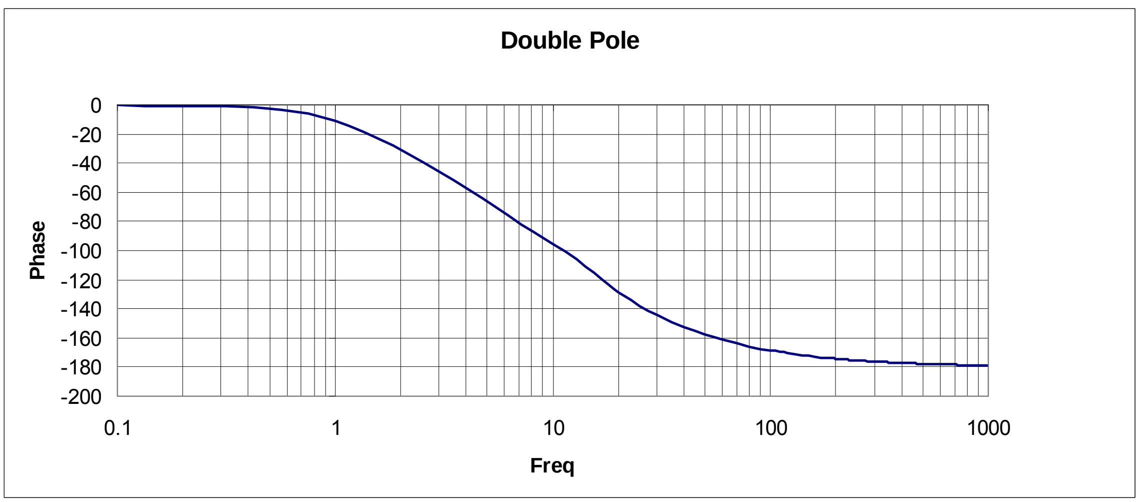
$$Magnitude(\omega) = 20\log(1) - 40\log\left(\sqrt{1+\frac{\omega^2}{10^2}}\right)$$

$$Phase(\omega) = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

Note the differences between this example and that of the single pole; the roll-off is twice as steep and the phase difference at any frequency is doubled.



Also notice that at the break frequency the magnitude is down by -6db, i.e., -3db for each pole.



Double Pole Hi-Pass

Consider the double pole hi-pass transfer function:

$$\frac{N(s)}{(s+\omega_o)^2}$$

In plotting format:

$$20\log\frac{Output}{Input} = 20\log N(s) - 40\log\left(\sqrt{1+\left(\frac{\omega}{\omega_o}\right)^2}\right)$$

$$Phase = \arctan N(s) - 2 * \arctan\left(\frac{\omega}{\omega_o}\right)$$

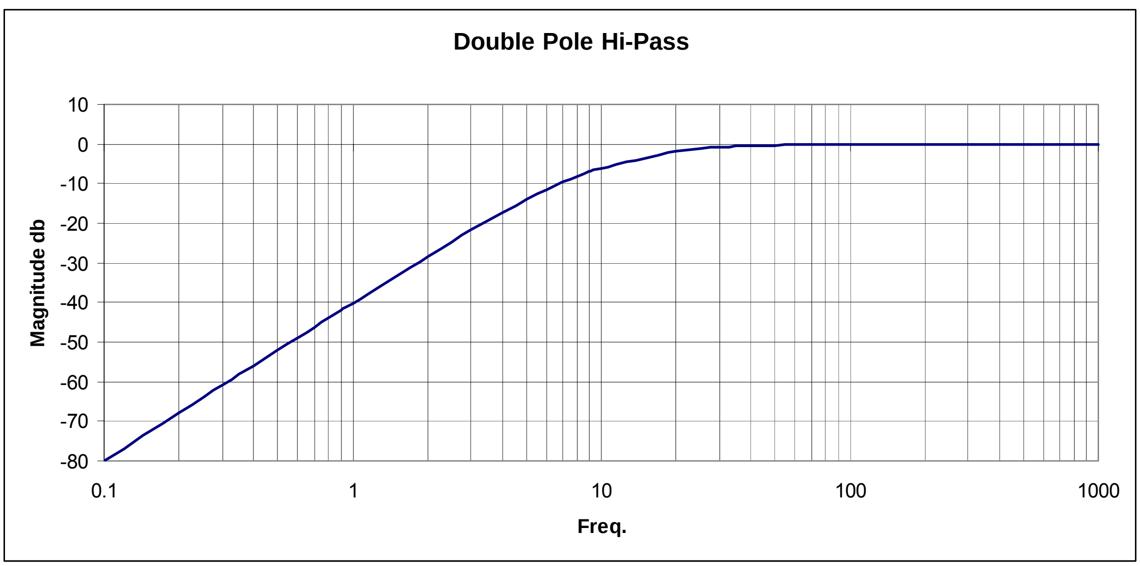
As an example:

$$\frac{s^2}{(s+10)^2}$$

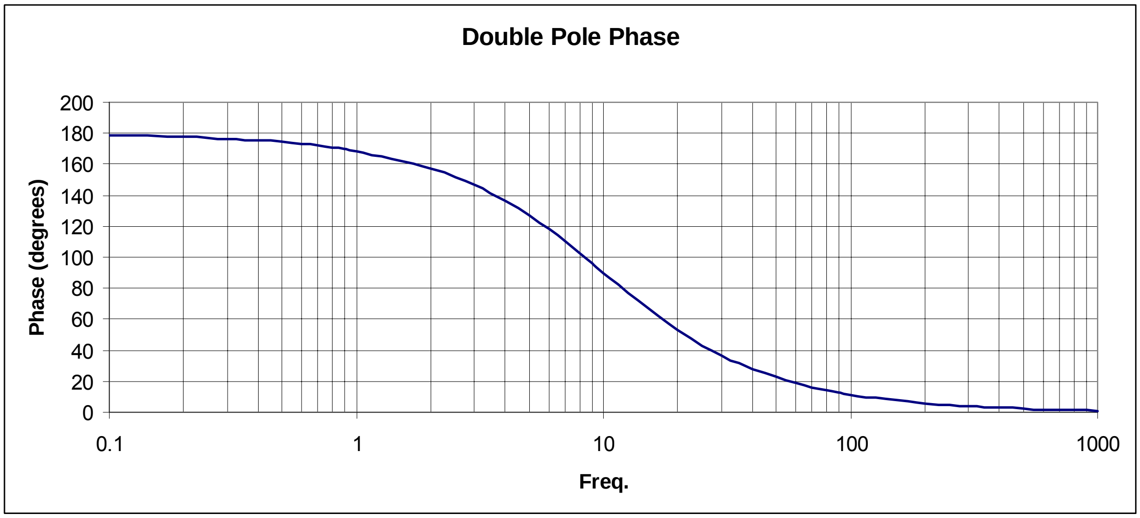
Then the plotting equations are:

$$Magnitude = 40\log\left(\frac{\omega}{10}\right) - 40\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right)$$

$$Phase = 180 - 2 * \tan^{-1}\left(\frac{\omega}{10}\right)$$



Since the definition of bandwidth (see discussion below) anchors on the -3db points, notice that it has shifted from the break frequency of 10rads/s to about 20 rads/s. As we shall see in subsequent examples, this shifting serves to tighten up or narrow the bandwidth in filters designed for bandpass selection.



Notice that at the break frequency the phase shift is 2*45^{o}.

There are design considerations in multiple pole filters that are not apparent in the transfer function. The practical reality of converting a transfer function into a physical circuit is not a straight forward one for one conversion from a LaPlace expression into a schematic. Techniques to accomplish that process are deferred until later modules. Some knowledge of active circuits will ease the transition from transfer function to schematic. Discussions of active circuits and LaPlace transforms are the focus of Module 5.

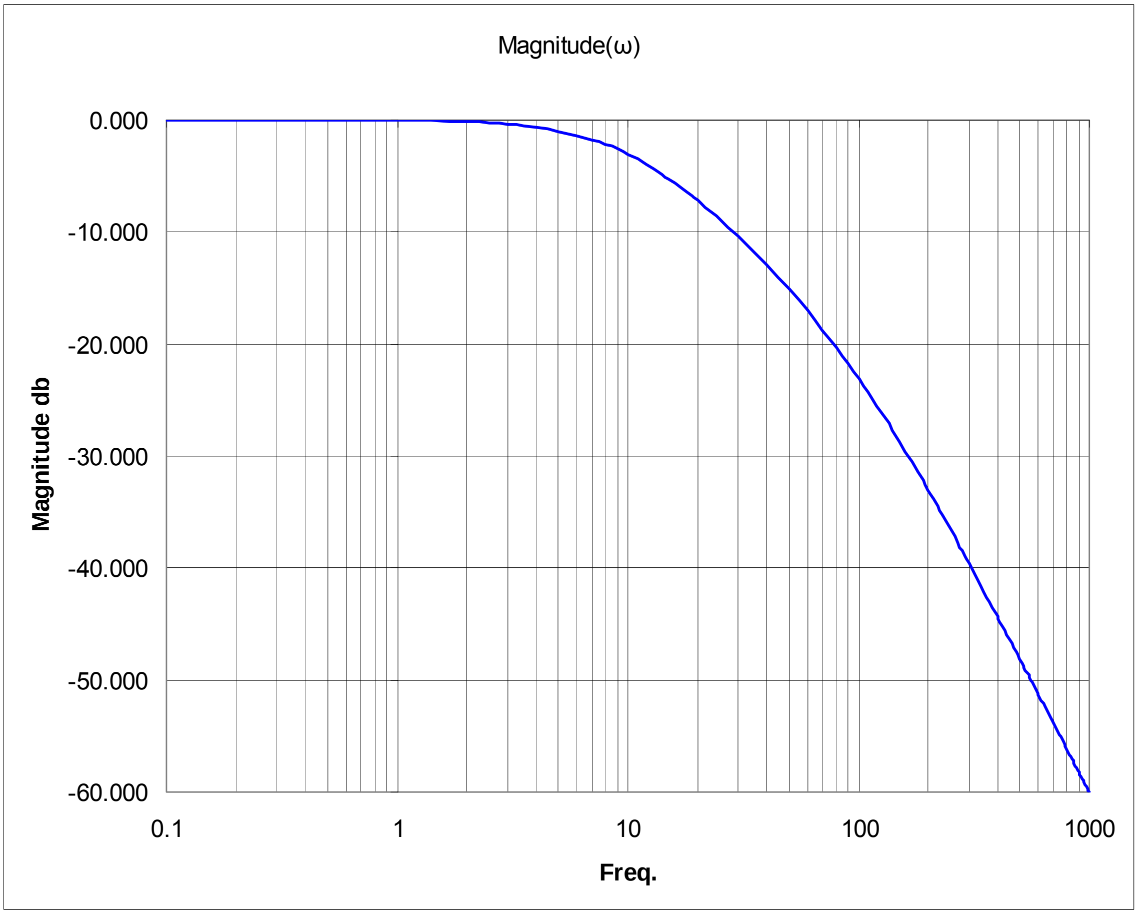
Next, consider an example of a double pole function with differing break frequencies:

$$\frac{Output}{Input} = \frac{1000}{(s+10)(s+100)} = \frac{1}{\left(1+\frac{s}{10}\right)\left(1+\frac{s`}{100}\right)}$$

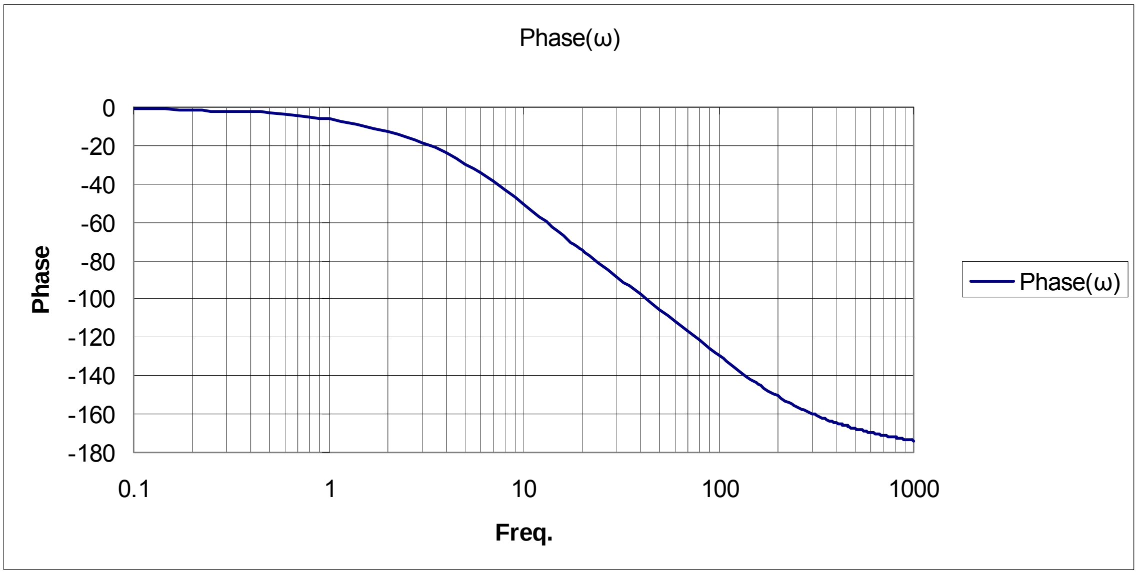
Forming the necessary equations for plotting:

$$Magnitude = 20\log(1) - 20\log\left(\sqrt{1+\frac{\omega^2}{10^2}}\right) - 20\log\left(\sqrt{1+\frac{\omega^2}{100^2}}\right)$$

$$Phase = \tan^{-1}(0) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{100}\right)$$



As expected, the roll-off between 10 and 100 rads/s s is -20db per decade, whereas the roll-off between 100 and 1,000 rads/s is -40db per decade. The pattern is that each pole contributes a -20db per decade roll-off beginning at its break frequency. If a -100db per decade roll-off is needed, then you will need a 5 pole filter. -Ndb per decade requires N/20 poles. While easy to understand, it is not as easy to implement; more on that topic in a later module.



Another pattern that is evident is that each pole contributes a -90^o phase shift at the output. N poles = N*(-90^o) shifts. That fact allows you to predict the terminal phase shift as $\omega \rightarrow \infty$.

Bandwidth and Half-Power Points

Notice that the magnitude is -3db at the break frequency ($\omega = \omega_o$). -3db is known as the half-power point since 10Log(.5) = -3. Filter bandwidth is usually defined as the range of frequencies between half-power points. In notch and bandpass filters, there will be a pair of half-power points; one for roll-on and one for roll-off. In the case of the lo-pass filter as above, there is only one half-power point. More will be mentioned of this topic later in the module.

Adding Zeros to the Transfer Function

Suppose there exists a transfer function such as:

$$\frac{K(s+\alpha)}{(s+\beta)}$$

Using the procedures we have already developed, the expressions necessary to plot magnitude vs. frequency and phase shift vs. frequency are:

$$Magnitude = 20\log(K) + 20\log\left(\sqrt{1+\frac{\omega^2}{\alpha^2}}\right) - 20\log\left(\sqrt{1+\frac{\omega^2}{\beta^2}}\right)$$

$$Phase = \tan^{-1}(0) + \tan^{-1}\left(\frac{\omega}{\alpha}\right) - \tan^{-1}\left(\frac{\omega}{\beta}\right)$$

For example:

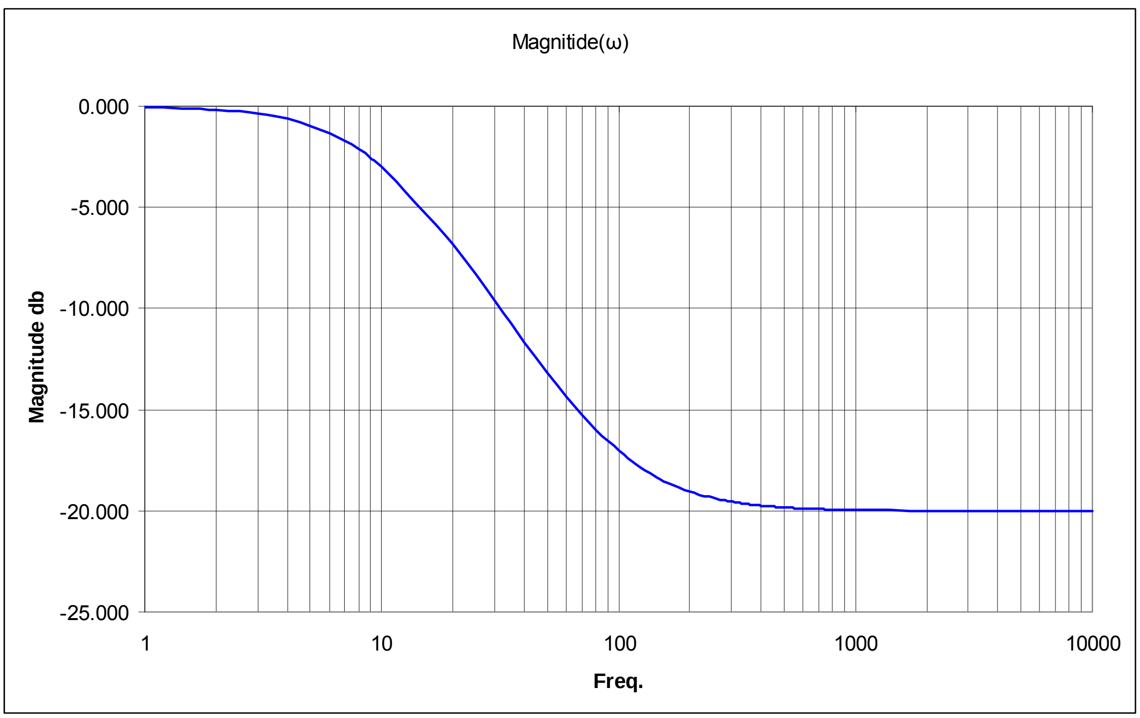
$$\frac{.1(s+100)}{(s+10)} = \frac{\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{10}\right)}$$

and then:

$$Magnitude = 20\log(1) + 20\log\left(\sqrt{1+\left(\frac{\omega}{100}\right)^2}\right) - 20\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right)$$

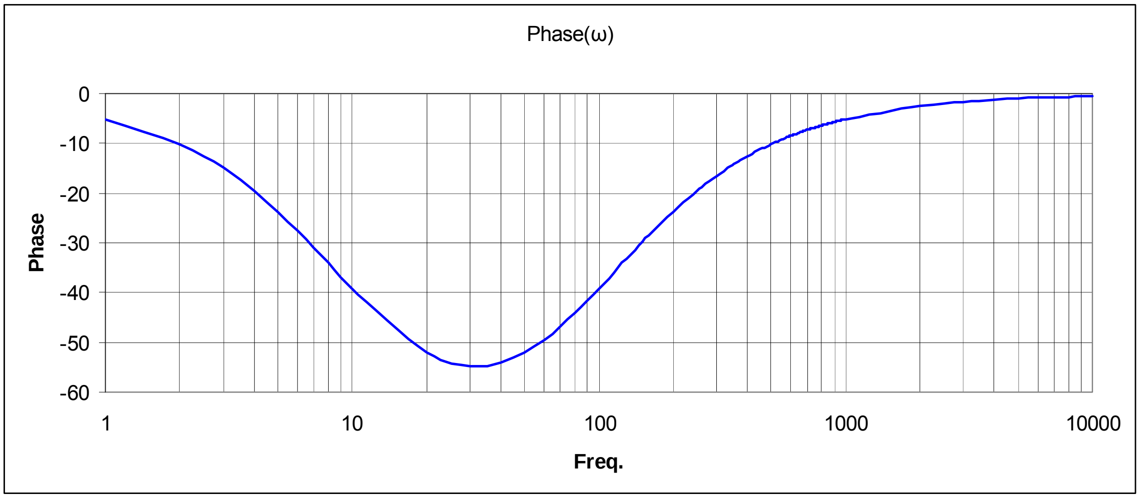
$$Phase = \tan^{-1}(0) + \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{10}\right)$$

Plotting these:



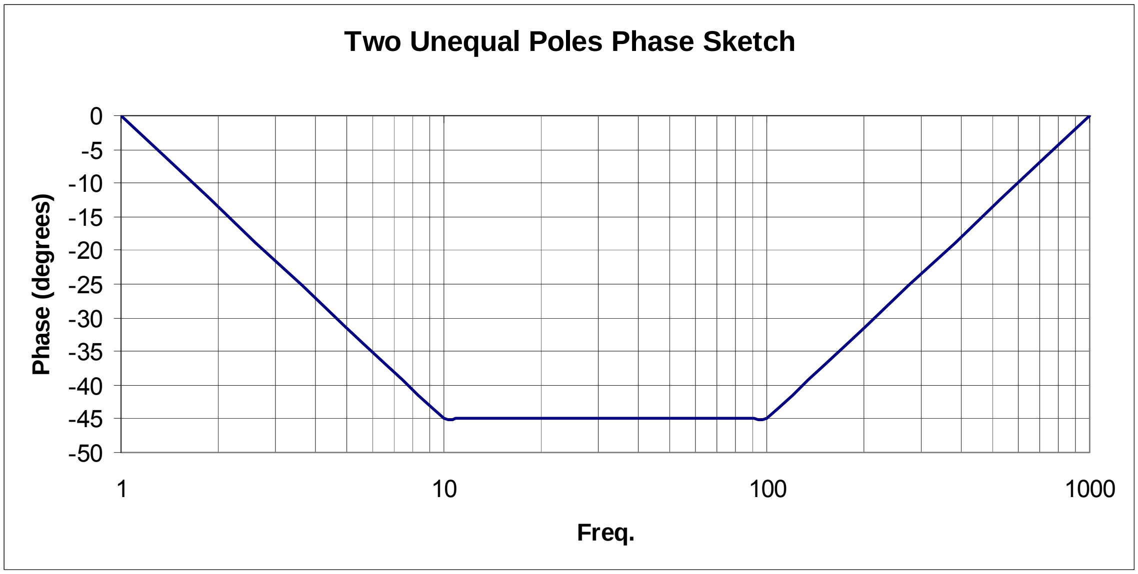
From the expression that the magnitude versus frequency plot is constructed, we should expect a -20db roll-off beginning at 10 rads/s and a +20db roll-on at 100 rads/s.

From the above graph those two effects can be seen. Since the +20db term cancels the -20db term beginning at 100 rad break frequency, the magnitude of the output with respect to the input remains constant beyond about 500 rads/s. The net effect of the zero is to cancel the effect of the pole at the zero's break frequency. This effect is very useful in designing and constructing bandpass and notch filters.



The above case illustrates the problems in determining the frequency at which changes occur if sketching a phase plot (try sketching it and predicting the phase between 10 and 1000 rads/s). The calculated plot is easy to construct and very accurate. It was simple to predict that the terminal phase change would sum to zero as $\omega \to \infty$, since $-90^o+90^o=0$. Since the two break frequencies are 10 rads/s and 100 rads/s we should expect maximum phase changes $\left(\frac{d\phi}{d\omega}\right)$ to occur within that range of frequencies. From the plot it is seen that there is indeed a minimum accompanied by an expected sign change.

Below is my attempt to sketch the phase. As can be seen, it has a general resemblance to the computed plot but lacks a great deal in accuracy. My preference is to rely on computed plots while visualizing the sketch for a first look.



Multiple Poles, Multiple Zeros

Next, consider the two zero-two pole bandpass filter whose general transfer function is:

$$\frac{Output}{Input} = \frac{K(s+\alpha)(s+\varepsilon)}{(s+\beta)(s+\varphi)}$$

In order to serve as a bandpass, $\alpha < \beta < \varphi < \varepsilon$. As an example (frequencies chosen for illustration convenience):

$$\frac{(s+10)(s+10000)}{(s+100)(s+1000)} = \frac{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{10000}\right)}{\left(1+\frac{s}{100}\right)\left(1+\frac{s}{1000}\right)}$$

Performing the necessary substitution:

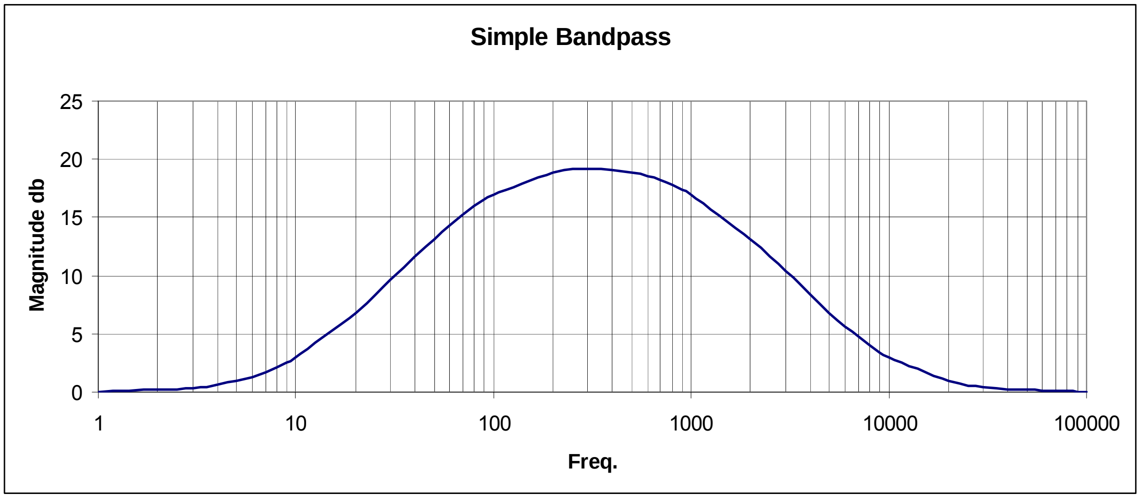
$$\frac{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{10000}\right)}{\left(1+\frac{j\omega}{100}\right)\left(1+\frac{j\omega}{1000}\right)}$$

The plotting equations are:

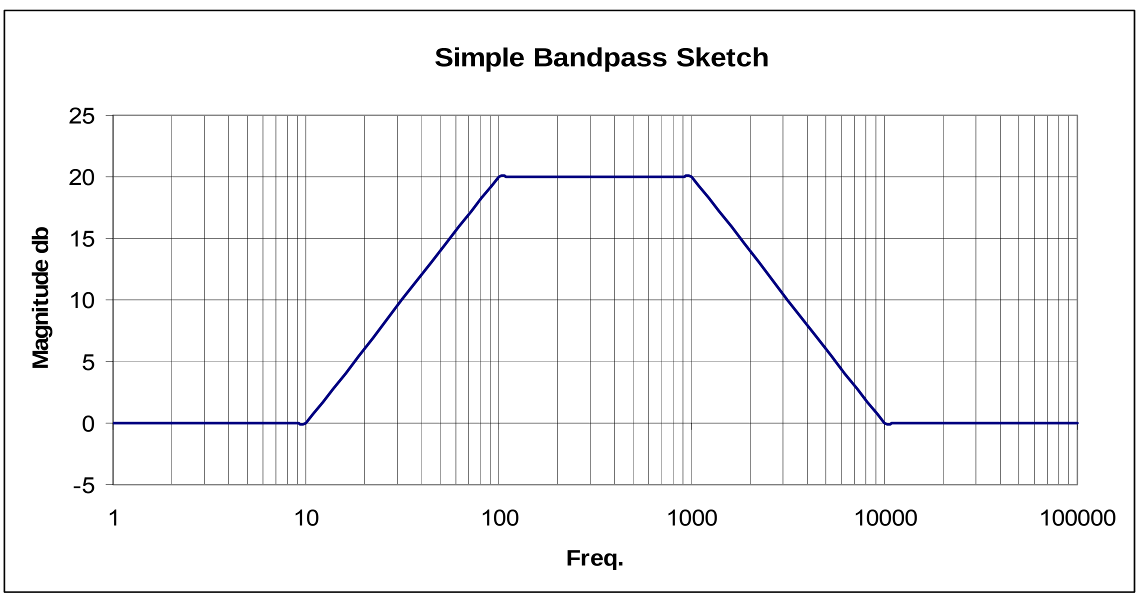
$$Mag = 20\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right) + 20\log\left(\sqrt{1+\left(\frac{\omega}{10000}\right)^2}\right) - 20\log\left(\sqrt{1+\left(\frac{\omega}{100}\right)^2}\right) - 20\log\left(\sqrt{1+\left(\frac{\omega}{1000}\right)^2}\right)$$

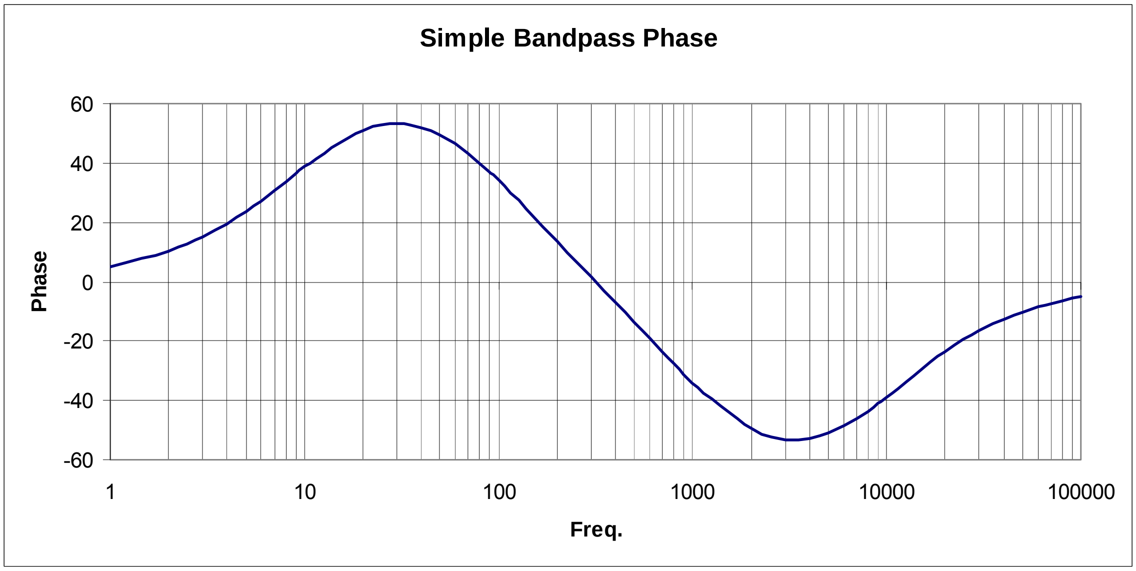
$$Phase = \tan^{-1}\left(\frac{\omega}{10}\right) + \tan^{-1}\left(\frac{\omega}{10000}\right) - \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}\left(\frac{\omega}{1000}\right)$$

Plotting these:



Above is the calculated plot, and below is the estimated or sketched plot. While the general form of the sketch conforms to the general locus of the calculated plot, it misses the point on the maximum magnitude and the 3db points. The bandpass of the above is about 85 to 1,020 rads/s. The bandpass below is about 70 to 1,060 rads/s.



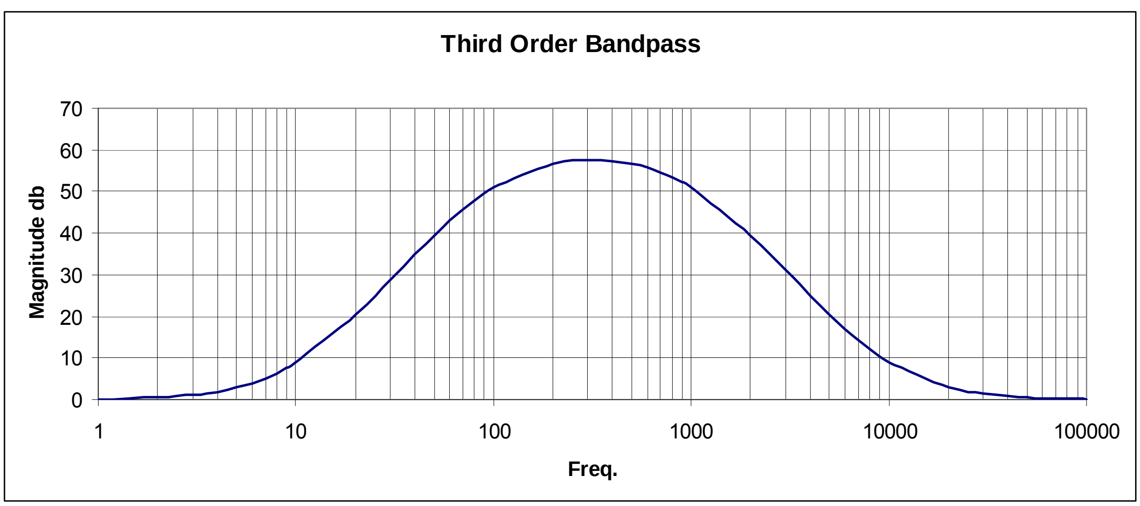


To illustrate the effect of multiples at the zeros and poles, let us modify the above filter to be a third order. The plotting equations are then:

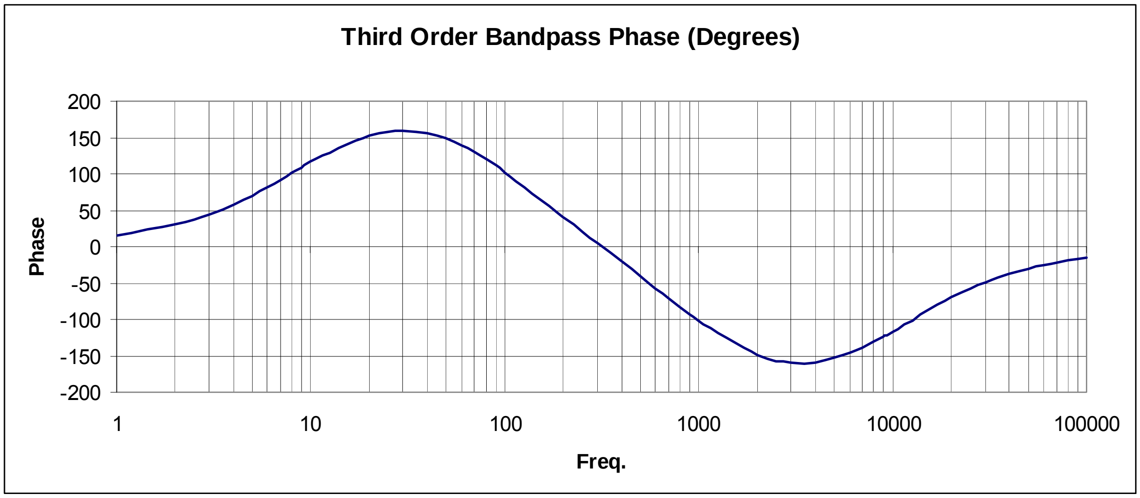
$$Mag. = 60\left(\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right)+\log\left(\sqrt{1+\left(\frac{\omega}{10^4}\right)^2}\right)-\log\left(\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}\right)-\log\left(\sqrt{1+\left(\frac{\omega}{10^3}\right)^2}\right)\right)$$

$$Phase = 3\left(\tan^{-1}\left(\frac{\omega}{10}\right)+\tan^{-1}\left(\frac{\omega}{10^4}\right)-\tan^{-1}\left(\frac{\omega}{10^2}\right)-\tan^{-1}\left(\frac{\omega}{10^3}\right)\right)$$

Plotting these:



Other than the change in magnitude, the most significant observation is that the bandpass went from 85-1,050 rads/s for the first order filter to approximately 150-700 rads/s; a tightening up of the bandwidth. The third order filter has about 57% of the bandwidth of the first order filter. We can tentatively conclude that increasing the order of a filter, decreases its bandwidth; thereby, increasing its selectivity.



The pattern of the third order is identical to the first order with the exception that the Y axis values have been multiplied by 3. The third order has a $\pm 60db$ per decade change in magnitude and a total potential phase change of $\pm 270^o$ (why does the computed plot fall well short of that value? What effect does the spacing of the break frequencies have on peak-to-peak phase excursion?). It is fairly easy to change the order of a filter in the model while the circuit is often a different matter. This will be addressed in subsequent modules.

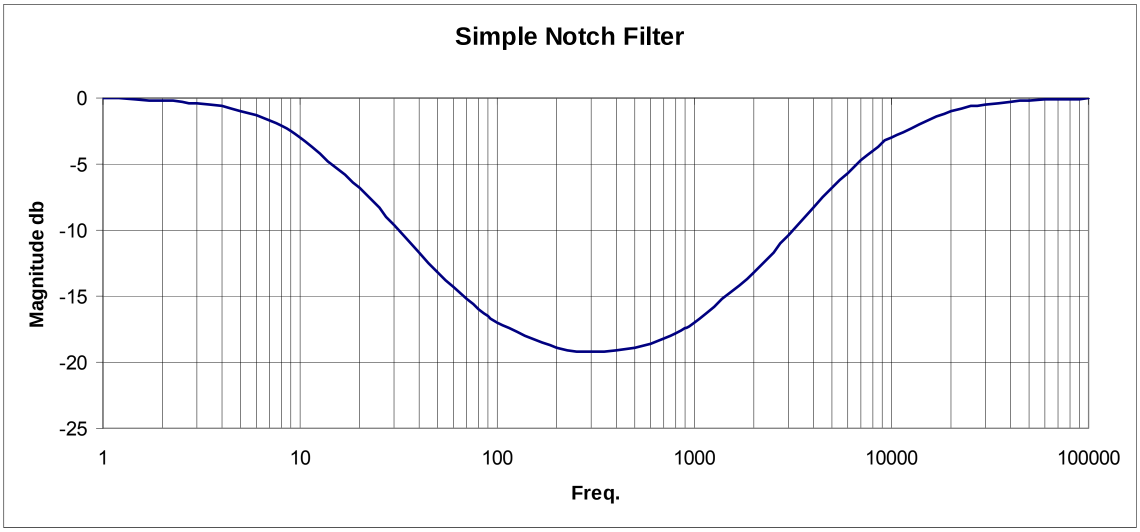
To create a notch filter, interchange the break frequency poles and zeros. For example:

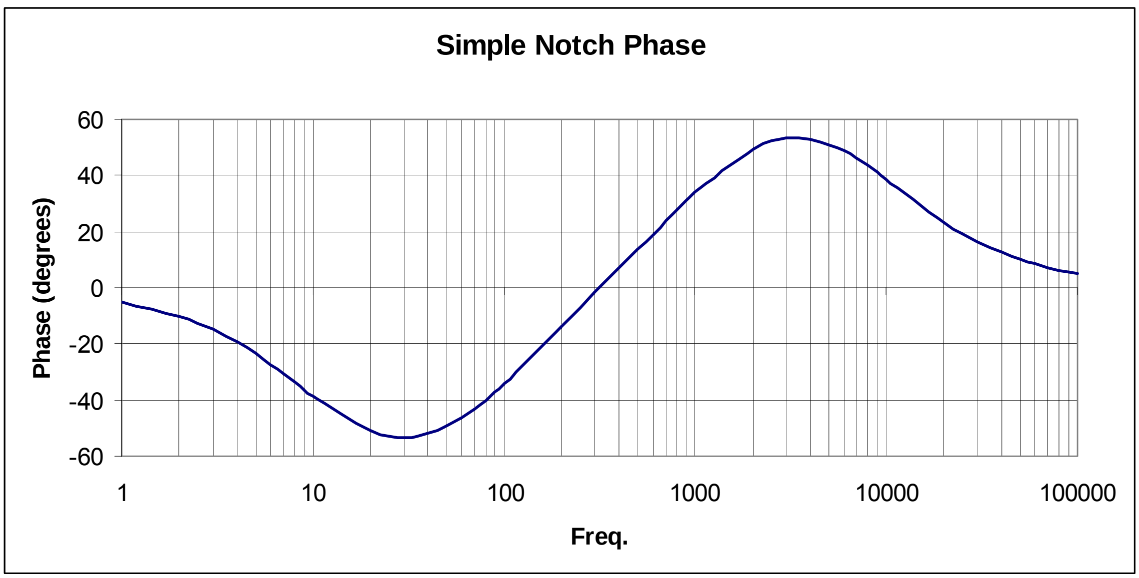
$$\frac{Output}{Input} = \frac{(s+100)(s+1000)}{(s+10)(s+10^4)}$$

$$Mag = 20\left(\log\left(\sqrt{1+\left(\frac{\omega}{10^2}\right)^2}\right) + \log\left(\sqrt{1+\left(\frac{\omega}{10^3}\right)^2}\right) - \log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right) - \log\left(\sqrt{1+\left(\frac{\omega}{10^4}\right)^2}\right)\right)$$

$$Phase = \tan^{-1}\left(\frac{\omega}{100}\right) + \tan^{-1}\left(\frac{\omega}{10^3}\right) - \tan\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{10^4}\right)$$

Plotting these:





Break frequency Spacing and Bandwidth

The effect of decreasing the spacing between the break frequencies will decrease the bandwidth, but the slope of ± 20 db/decade per pole/zero remains unchanged. As a consequence, peak magnitude is correspondingly decreased. For example:

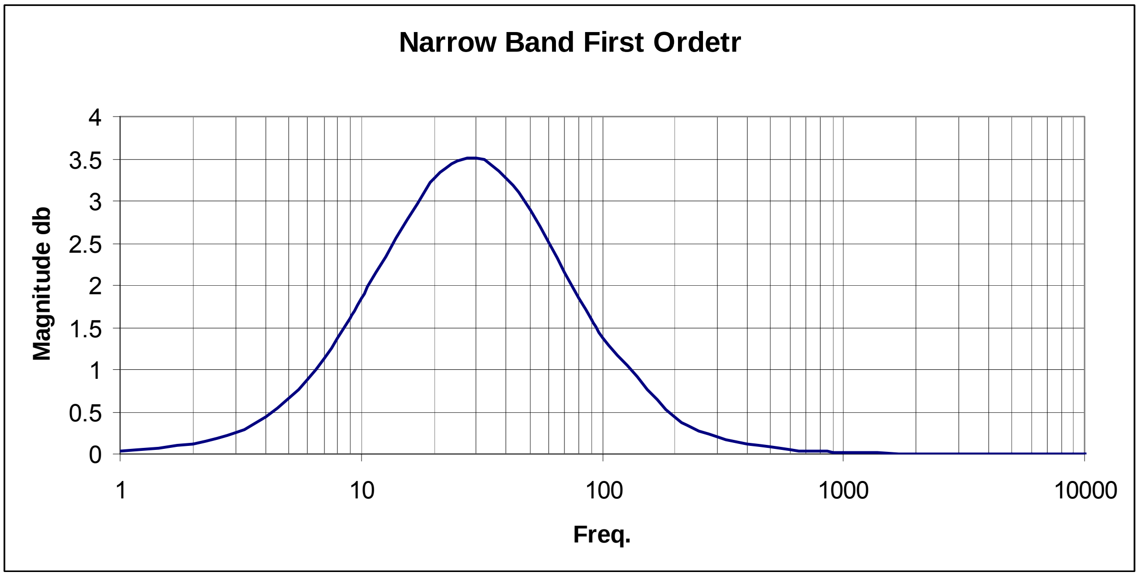
$$\frac{Output}{Input} = \frac{(s+10)(s+80)}{(s+20)(s+40)}$$

The plotting equations are:

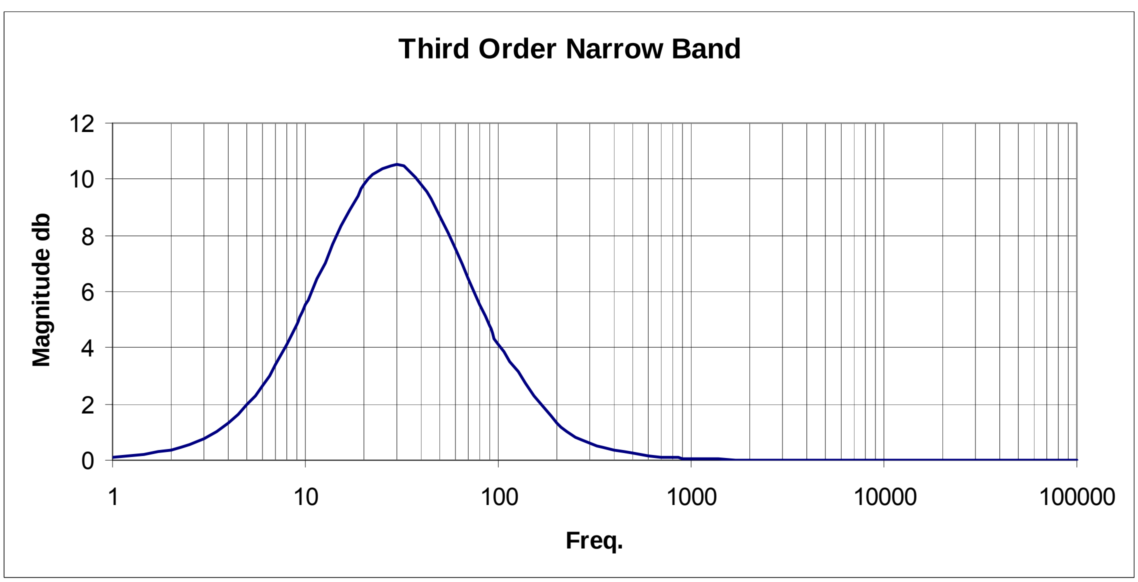
$$Mag = 20(\log\left(\sqrt{1+\left(\frac{\omega}{10}\right)^2}\right) + \log\left(\sqrt{1+\left(\frac{\omega}{80}\right)^2}\right) - \log\left(\sqrt{1+\left(\frac{\omega}{20}\right)^2}\right) - \log\left(\sqrt{1+\left(\frac{\omega}{40}\right)^2}\right)$$

$$Phase = \tan^{-1}\left(\frac{\omega}{10}\right) + \tan^{-1}\left(\frac{\omega}{80}\right) - \tan^{-1}\left(\frac{\omega}{20}\right) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

These equations plot to:



The bandwidth has been decreased to approximately 4-119 rads/s. Increasing the order of the transfer function to 3, produces:



This decreases the bandwidth to about 45 rads/s. The gain is both magnitude and selectivity.

The Complex Quadratic Pole

Consider a transfer function such as:

$$\frac{Output}{Input} = \frac{N(s)}{s^2 + \frac{b}{a}s + \frac{c}{a}}$$

where $\frac{b}{a}$ & $\frac{c}{a}$ are such that the roots are complex. Let us change the notation to something more convenient to our purposes; let $\frac{b}{a} = 2\zeta\omega_0$ and $\frac{c}{a} = \omega_o^2$, then:

$$\frac{Output}{Input} = \frac{N(s)}{s^2 + 2\zeta\omega_o s + \omega_o^2}$$

In order to remain complex, restrict $0 \le \zeta < 1$.

Substituting $j\omega$ for s:

$$\frac{Output}{Input} = \frac{N(s)}{\left(\omega_o^2 - \omega^2\right) + j2\zeta\omega_o\omega}$$

$$\frac{Output}{Input} = \frac{N'(s)}{\left(1 - \frac{\omega^2}{\omega_o^2}\right) + j2\zeta\frac{\omega}{\omega_o}}$$

As an example, set $N'(s) = 1$ and $\omega_o = 1$. Since the salient effect occurs as a function of ζ (zeta), plots will be made for a few different values of ζ.

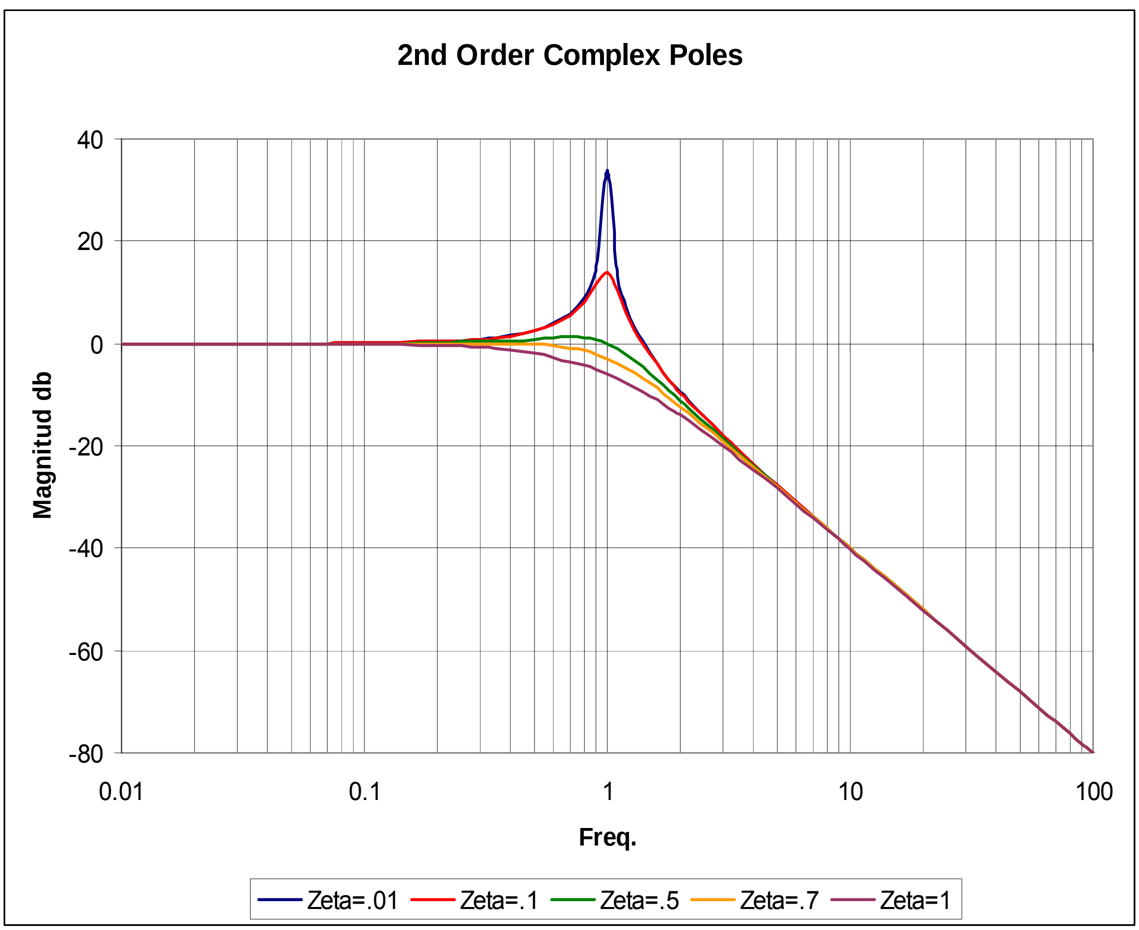
$$Mag. = 20\log(1) - 20\log\left(\sqrt{\left(1 - \frac{\omega^2}{1}\right)^2 + \left(2\zeta\frac{\omega}{1}\right)^2}\right)$$

$$Phase = 0 - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_o}}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$

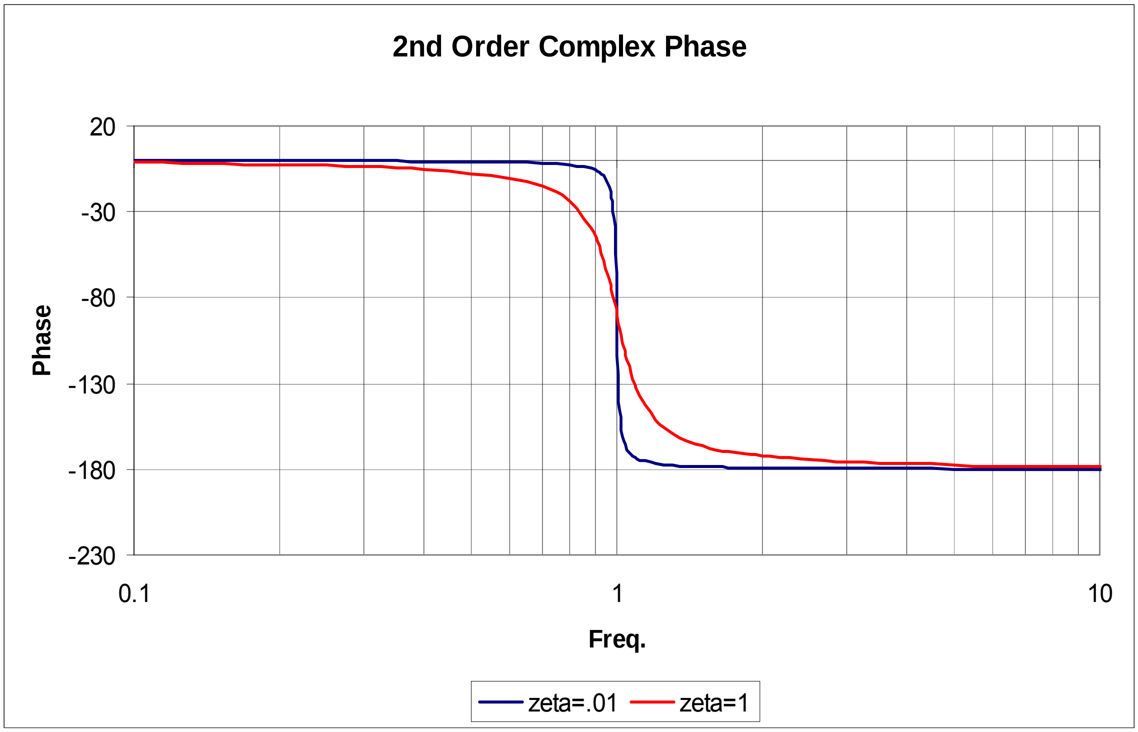
Use the above when the denominator is positive. Remember at ω_o the phase is -90^o; thereby, avoiding division by zero.

$$Phase = -180 - \tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_o}}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}\right)$$

Use the above when the denominator is negative.



We still get the -40 db/decade roll-off, but the response of the system about the break frequency varies significantly as a function of zeta. When zeta=.01, there exists an extremely narrow selectivity bandwidth about the break frequency. When zeta=.01, the time constant, in this case, equals 100s and the system impulse response dies out at 600s. When $\zeta = 1$, the system is just a 2nd order repeated real pole filter at a break frequency of 1.



When the resonant frequency is changed, all that changes in the pattern is the frequency that the peaks occur at in the magnitude plot, and the frequency that holds the -90^{o} spot in the phase plot.

For example:

Since we have used the notation $s^2 + 2\zeta\omega_o s + \omega_o^2$, and since we restrict this notation to those cases wherein the roots are complex, it is legitimate to ask about the relationship between the resonant frequency (ω_o) and the damped frequency (ω_d) as a function of ζ (we will label ζ the damping coefficient from here on out to the end of the series of modules). Clearly when $\zeta = 0$, the equation becomes $s^2 + \omega_o^2$ and the roots are $\pm j\omega_o$; that is the resonant frequency and the frequency of oscillation. However, when $0 < \zeta < 1$, then the roots of the equation are $-\zeta\omega_o \pm j\sqrt{\omega_o^2 - (\zeta\omega_o)^2}$, where the frequency of oscillation $\omega_d = \sqrt{\omega_o^2 - (\zeta\omega_o)^2}$ or

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

To verify, let a denominator be $s^2 + 4s + 13$. The roots are $s = -2 \pm j3$, $\omega_o = \sqrt{13}$, $\omega_d = 3$ and $4 = 2\zeta\omega_o$; therefore $\zeta = \frac{2}{\sqrt{13}}$. Therefore:

$$3 = \sqrt{13}\sqrt{1-\left(\frac{2}{\sqrt{13}}\right)^2} = \sqrt{13-4} = \pm 3 \quad \text{Q.E.D.}$$