LaPlace Transforms in Design and Analysis of Circuits

Part 3: Basic Parallel Circuit Analysis

Course No: E04-017

Credit: 4 PDH

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LaPlace Transforms in Design and Analysis of Circuits©
Part 3

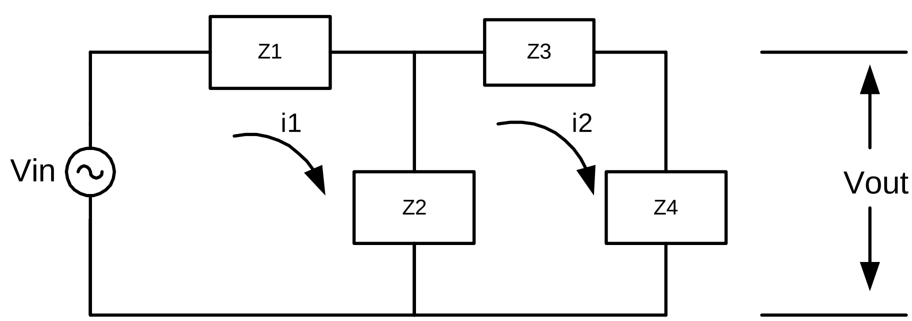
by Tom Bertenshaw

Basic Circuit Analysis - Parallel Circuits

Simple Two Loop

In Part 2, LaPlace techniques were used to solve for the output in simple series reactive circuits. This part will examine the techniques used in approaching the solution to two and three loop parallel circuits with reactive components. Parallel circuits that contain a number of loops beyond three will not be exampled, as any number of loops can be reduced to two or three by the use of Thevinan's or Norton's theorem, or by merely extending the procedures developed here (although the math housekeeping gets intense).

Consider the following two loop system:



Writing the circuit equation for each loop; reveals that there are two equations in two unknowns that characterize the circuit.

$$V_{in} = (z_1 + z_2)i_1 - z_2 i_2$$
$$0 = -z_2 i_1 + (z_2 + z_3 + z_4)i_2$$

There are several techniques available for solving systems of linear equations that have the same number of equations as unknowns, and of those, we will use Cramer's Rule throughout this section (If you want a quick refresher on the use of Cramer's Rule for solving systems of linear equations, see Appendix A). Suppose we have the following values for z_1 through z_4

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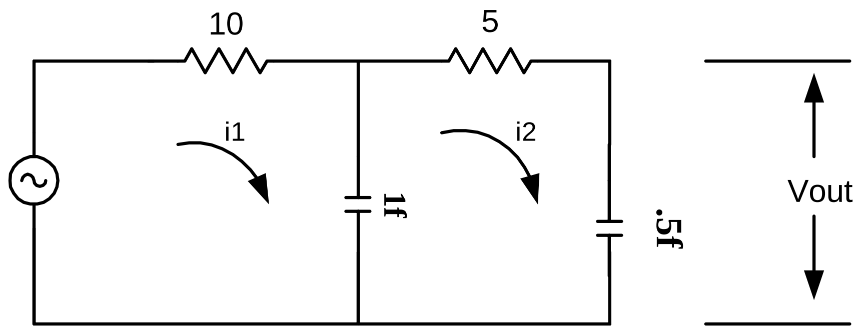


Circuit 1

Writing the equations for the two loops;

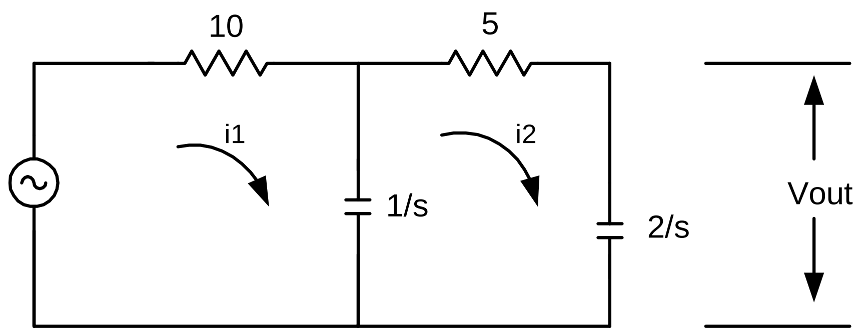
$$V_{in} = 10i_1 + \int (i_1 - i_2) dt$$
$$0 = -\int i_1 dt + 5i_2 + 3\int i_2 dt$$

Question: How was the factor $3\int i_2 dt$ developed?

Since the defining equation for capacitor behavior is $i_c = C\frac{dv_c}{dt}$, it follows that

$v_c = \frac{1}{C}\int i_c dt$.

We can probably reduce the labor (and anguish) if the above two circuit equations are converted to their LaPlace transforms. Looking at the transformed circuit;



the circuit equations are re-written as;

$$V_{in}(s) = \left(10 + \frac{1}{s}\right) i_1(s) - \frac{1}{s} i_2(s)$$
$$0 = -\frac{1}{s} i_1(s) + \left(5 + \frac{3}{s}\right) i_2(s)$$

Re-writing these equations as matrices;

$$\begin{bmatrix} \left(10+\frac{1}{s}\right) & -\frac{1}{s} \\ -\frac{1}{s} & \left(5+\frac{3}{s}\right) \end{bmatrix} \begin{bmatrix} i_1(s) \\ i_2(s) \end{bmatrix} = \begin{bmatrix} V_{in}(s) \\ 0 \end{bmatrix}$$

First, find the determinant.

$$Det. = \left(10+\frac{1}{s}\right)\left(5+\frac{3}{s}\right) - \frac{1}{s^2} = \frac{50s^2 35s+2}{s^2} = \frac{50\left(s^2+.7s+.04\right)}{s^2}$$

The task is to find V_{out} as a function of the driver, and in this case let $V_{out}(s) = i_2(s)\,2/s$ (output taken across rightmost capacitor).

However, first we'll find the impulse response to gain an idea of the form of the transient response of the circuit. Because $i_2(s)$ is the only current involved in finding $V_{out}(s)$ we need not solve for $i_1(s)$, except to satisfy curiosity. Solving for $i_2(s)$;

$$\frac{\begin{bmatrix} \left(10+\frac{1}{s}\right) & 1 \\ -\frac{1}{s} & 0 \end{bmatrix}}{Det.} = \frac{s}{50\left(s^2+.7s+.04\right)}$$

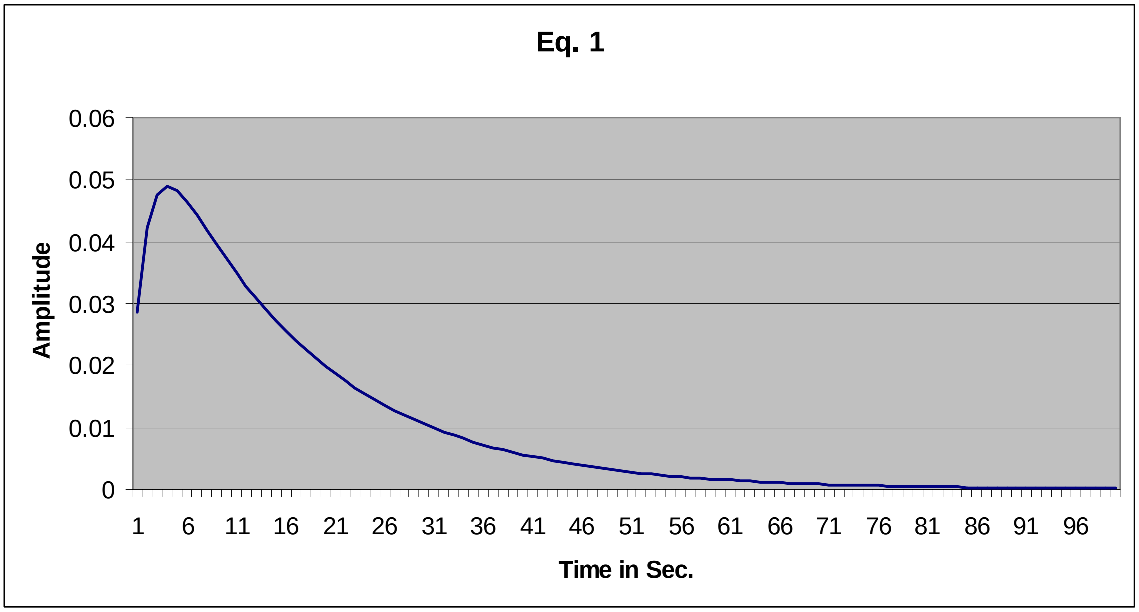
and the output voltage response to a current impulse is

$$V_{out}(s) = \left(\frac{s}{50\left(s^2+.7s+.04\right)}\right)\left(\frac{2}{s}\right) \approx .07\left(\frac{-1}{s+.637} + \frac{1}{s+.063}\right)$$

in turn then

$$V_{out}(t) \approx .07\left(e^{-.063t} - e^{-.637t}\right) \quad \leftarrow \text{Ex. 1}$$

Indicating that the transient response, for all practical purposes, is zero at $t \approx 95$ seconds (recall that in previous modules e^{-6} was defined to be about zero).



Suppose that a driver of $1\sin(.1t)$ is applied as $V_{in}(t)$. (Often parameters are chosen, in this case the frequency, to exaggerate some aspect of the response for illustration purposes.)

Then $i_2(s)$ equals

$$\frac{\begin{bmatrix} \left(10+\frac{1}{s}\right) & \frac{.1}{s^2+.1^2} \\ -\frac{1}{s} & 0 \end{bmatrix}}{Det.} = \left(\frac{\left(\frac{1}{s}\right)\left(\frac{.1}{s^2+.01}\right)s^2}{50\left(s^2+.7s+.04\right)}\right) = \frac{.002s}{(s^2+.7s+.04)\left(s^2+.1^2\right)}$$

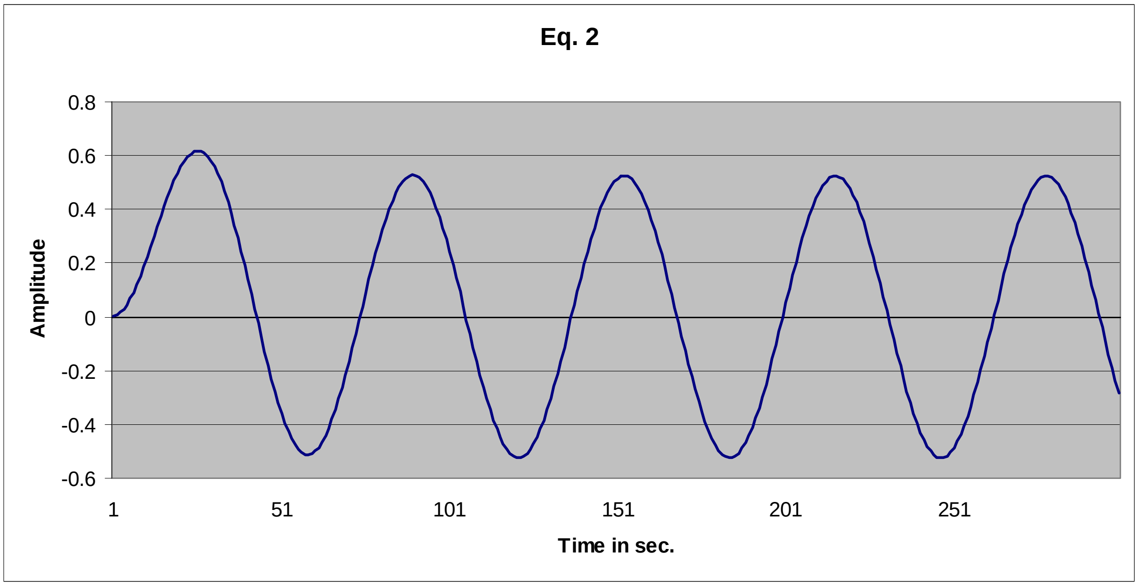
Since $V_{out}(s) = i_2(s)\left(\frac{2}{s}\right)$, then

$$V_{out}(s) \approx \frac{.004}{\left(s^2+.7s+.04\right)\left(s^2+.01\right)} \approx \frac{.5}{s+.063} - \frac{.017}{s+.637} + \frac{.525\angle -66^o}{s^2+.1^2}$$

(if you need a refresher on Partial Fraction Expansion, particularly with respect to finding factors on complex denominators, refer to LaPlace Transforms in Design and Analysis of Circuits© Part 2)

and finally

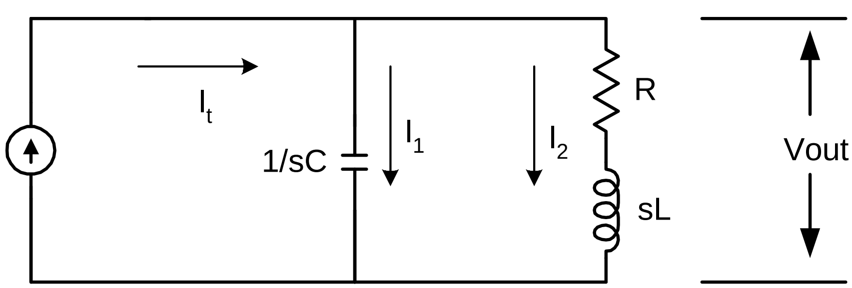
$$V_{out}(t) = .5e^{-.063t} - .017e^{-.637t} + .525\sin(.1t - 66^o) \qquad \leftarrow \text{Ex. 2}$$



Again, please note that the output contains two distinct components: the transient response and the steady state response. The transient response is the time limited contribution of the "native" or impulse response (amplitude modified by the nature of the driver). The steady state response has the characteristics of the driver and continues until the drive is removed. All responses will contain these two components.

The RLC

Another, and very significant, circuit is the analog of the series RLC network; as expected it is a parallel RLC network; often known as a "parallel tank" circuit. Its properties are such that it presents a very high impedance at the resonant frequency rendering the circuit very useful in filtering and frequency determination applications. Like other classic circuits this one can also be implemented using active components. However it is an understanding of the response that is sought at this time and not the techniques involved in mimicking reactive components with active components. Those design techniques will be developed in subsequent modules within this series.



Circuit 2

$$i_t = i_1 + i_2$$

by inspection,

$$i_1 = sCV_c$$

$$i_2 = \frac{V_{out}}{sL + R}$$

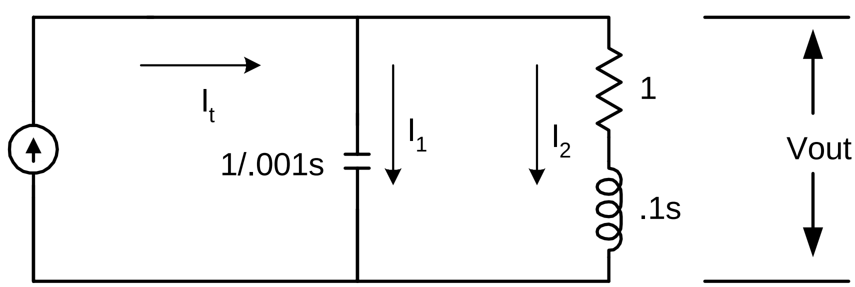
Since $V_c = V_{out}$, we can take advantage of that and write the circuit equation as

$$i_t = \left(sC + \frac{1}{sL + R} \right) V_{out}$$

or

$$\frac{V_{out}}{i_t} = \frac{sL + R}{s^2 LC + sCR + 1} = \frac{sL + R}{LC\left(s^2 + \frac{R}{L} s + \frac{1}{LC} \right)}$$

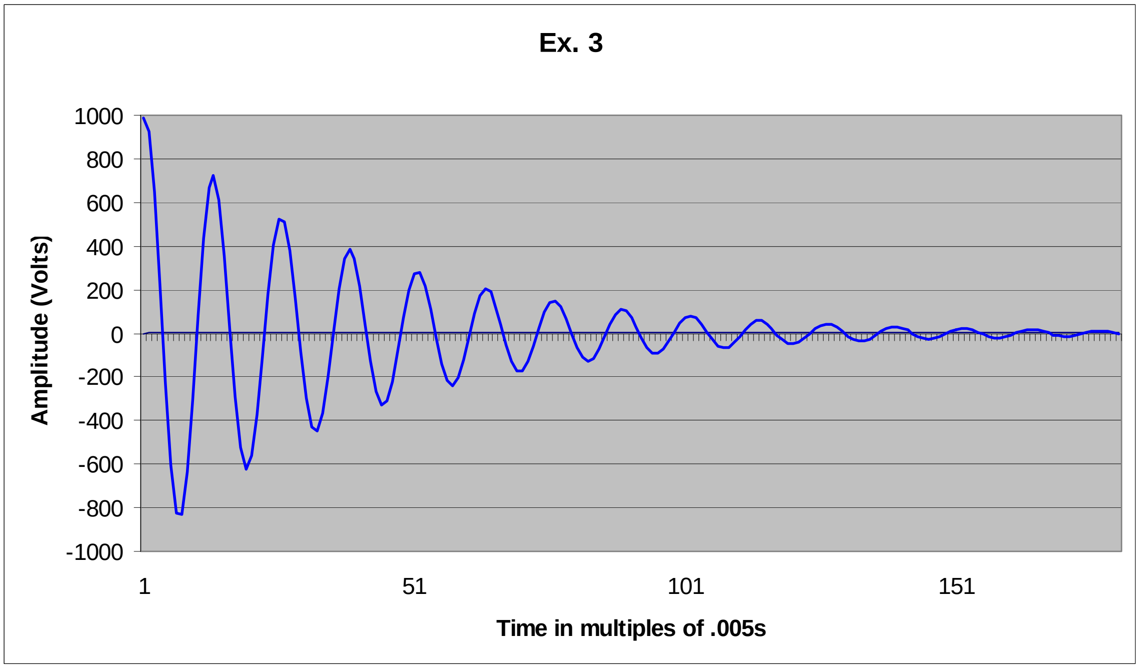
Suppose the following circuits exists, and that we wish to know its impulse response. (See Appendix C for a simple but relatively effective current driver)



Putting some flesh to the transfer function

$$V_{out}(s) = \frac{10000(.1s + 1)}{s^2 + 10s + 10000} \approx \frac{1000(s + 5 + 5)}{(s + 5)^2 + 99.87^2} = \frac{1000(s + 5)}{(s + 5)^2 + 99.87^2} + \frac{(99.87)(50.07)}{(s + 5)^2 + 99.87^2}$$

$$V_{out}(t) \approx 1000e^{-5t} \cos(99.87t) + 50.07e^{-5t} \sin(99.87t) \approx 1001e^{-5t} \sin(99.87t + 87^o) \leftarrow \text{Ex 3}$$



The tank exhibits a characteristic "ringing" that deteriorates over time in accordance with the system time constant. Task: identify the two components that determine the time constant; what is the theoretical result of a 0 ohm resistor? Question: is 0 ohm's possible under normal ambient conditions?

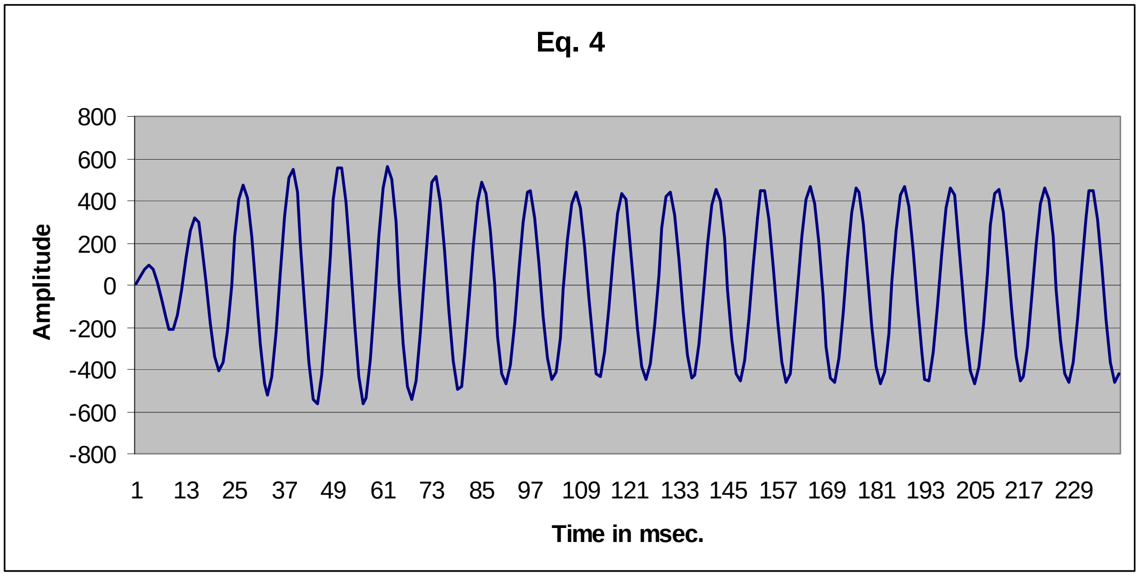
It is worth noting the magnitude of the output in response to $\delta(t)$. Parallel tanks generate considerable voltage at or very near their resonant frequency and, as a designer, one must always bear in mind the consequences to the remaining circuitry. It is a question of the physics of an inductor: Assume a total circuit current $i_t = A_Z \sin(\omega t)$ where the magnitude of A_Z is a function of total circuit impedance $\left(\frac{V_{in}}{Z_{total}} = i_t\right)$. At resonance, $jX_L - jX_C = 0$, and we also know that $V_L = L\frac{di}{dt}$; why is V_L so large?

Suppose this circuit is driven by $i_t = 10\sin(110t)$, then

$$V_{out}(s) = \frac{10000(1100)(.1s+1)}{(s^2+10s+10000)(s^2+110^2)} \approx \frac{(99.87)(464\angle 112^o)}{(s+5)^2+99.87^2} + \frac{(110)(465\angle -68^o)}{s^2+110^2}$$

$$V_{out}(t) \approx 464e^{-5t}\sin(99.87t+112^o) + 465\sin(110t-68^o) \quad \leftarrow \text{Eq. 4}$$

It would probably be prudent to reduce the total current input to .1 instead of a driving amplitude of 10.

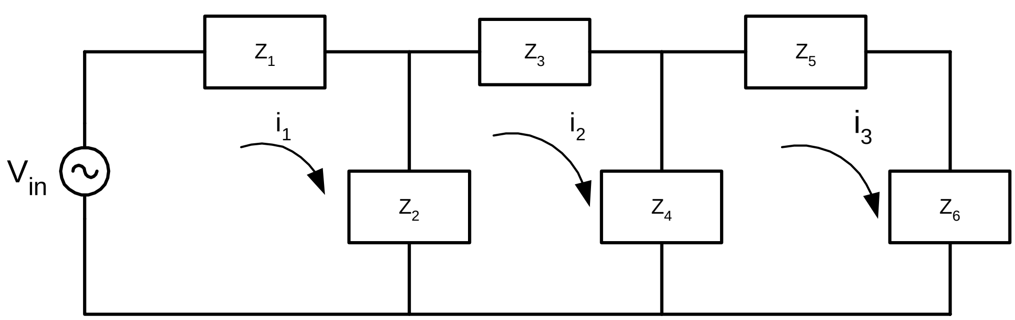


Bear in mind, from above, that $V_L = L\frac{di}{dt} = i_L(j\omega L)$, meaning that the magnitude of V_L is directly proportional to the magnitudes of both i_L & ω. The gross effect is that considerable voltage can be built across an inductor, which may present a hazard to the remaining circuitry. Of course there are techniques to control and/or limit the magnitudes, but that discussion is for a later time.

As an aside, and as a general comment: while considerable effort is maintained to monitor the correctness of all the calculations, oft times what can go wrong will go wrong. Therefore, if you discover an error, please do not hesitate to contact the company and/or the author.

Three Loop Circuit

Consider the following circuit



Circuit 3

Writing the loop equations

$$
\begin{aligned}
V_{in} &= (z_1 + z_2)i_1 - z_2 i_2 + 0i_3 \\
0 &= -z_2 i_1 + (z_2 + z_3 + z_4)i_2 - z_4 i_3 \\
0 &= 0i_1 - z_4 i_2 + (z_4 + z_5 + z_6)i_3
\end{aligned}
$$

Writing the system equations in matrix notation

$$
\begin{bmatrix} (z_1 + z_2) & -z_2 & 0 \\ -z_2 & (z_2 + z_3 + z_4) & -z_4 \\ 0 & -z_4 & (z_4 + z_5 + z_6) \end{bmatrix}
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} V_{in} \\ 0 \\ 0 \end{bmatrix}
$$

Writing the determinant using the either the example or the definition contained in Appendix A

$$
\begin{aligned}
Det. = &(z_1 + z_2)(z_2 + z_3 + z_4)(z_4 + z_5 + z_6) + (-z_2)(-z_4)(0) + (0)(-z_2)(-z_4) \\
&- (0)(z_2 + z_3 + z_4)(0) - (z_1 + z_2)(z_4^2) - (z_2^2)(z_4 + z_5 + z_6)
\end{aligned}
$$

$$
\begin{aligned}
Det = &z_1 z_2 z_4 + z_1 z_2 z_5 + z_1 z_2 z_6 + z_1 z_3 z_4 + z_1 z_3 z_5 + z_1 z_3 z_6 + z_1 z_4 z_5 + z_1 z_4 z_6 + z_2 z_3 z_4 + \\
&z_2 z_3 z_5 + z_2 z_3^{z} z_6 + z_2 z_4 z_5 + z_2 z_4 z_6
\end{aligned}
$$

Solving for i_1 through i_3

$$
i_1 = \frac{\begin{bmatrix} V_{in} & -z_2 & 0 \\ 0 & (z_2 + z_3 + z_4) & -z_4 \\ 0 & -z_4 & (z_4 + z_5 + z_6) \end{bmatrix}}{Det.} = \frac{V_{in}(z_2 + z_3 + z_4)(z_4 + z_5 + z_6) + 0 + 0 - 0 - V_{in}(z_4)^2 - 0}{Det.}
$$

$$
i_1 = V_{in}\left(\frac{z_2 z_4 + z_2 z_5 + z_2 z_6 + z_3 z_4 + z_3 z_5 + z_3 z_6 + z_4 z_5 + z_4 z_6}{Det.} \right)
$$

$$
i_2 = \frac{\begin{bmatrix} (z_1 + z_2) & V_{in} & 0 \\ -z_2 & 0 & -z_4 \\ 0 & 0 & (z_4 + z_5 + z_6) \end{bmatrix}}{Det.} = \frac{0 + 0 + 0 - 0 - 0 + V_{in} z_2 (z_4 + z_5 + z_6)}{Det.}
$$

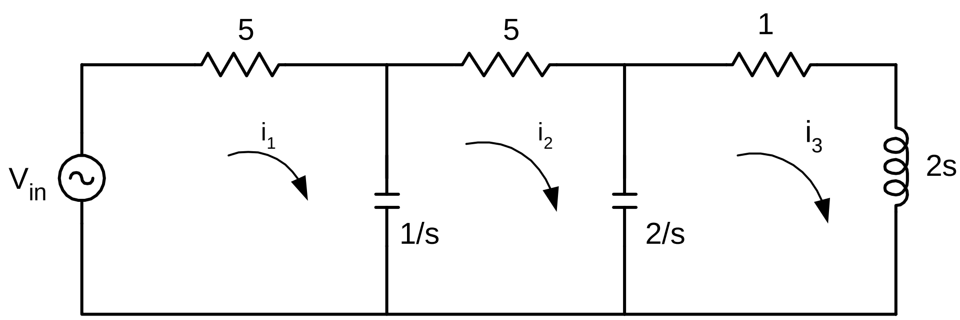
$$
i_2 = V_{in}\left(\frac{z_2 z_4 + z_2 z_5 + z_2 z_6}{Det.} \right)
$$

$$i_3 = \frac{\begin{bmatrix} (z_1+z_2) & -z_2 & V_{in} \\ -z_2 & (z_2+z_3+z_4) & 0 \\ 0 & -z_4 & 0 \end{bmatrix}}{Det.} = \frac{0+0+V_{in}z_2z_4 - 0 - 0 - 0}{Det.}$$

$$i_3 = V_{in}\left(\frac{z_2z_4}{Det.}\right)$$

Three by three matrices get messy when there are reactive components in the circuit, and even messier at greater dimensions. But as long as the fundamentals of the matrix solution process are understood, it is recommended that you resort to the use of a computer or calculator for solutions to systems greater than 3X3. Naturally, those aides are not essential, merely convenient.

For practice, consider the following circuit



The determinant matrix is

$$\begin{bmatrix} \left(5+\frac{1}{s}\right) & -\frac{1}{s} & 0 \\ -\frac{1}{s} & \left(5+\frac{3}{s}\right) & -\frac{2}{s} \\ 0 & -\frac{2}{s} & \left(1+2s+\frac{2}{s}\right) \end{bmatrix} =$$

$$\left(5+\frac{1}{s}\right)\left(5+\frac{3}{s}\right)\left(1+2s+\frac{2}{s}\right)+0+0-0-\left(5+\frac{1}{s}\right)\left(\frac{4}{s^2}\right)-\left(\frac{1}{s^2}\right)\left(1+2s+\frac{2}{s}\right)$$

collecting terms and factoring (courtesy of trusty HP49 - I cannot be sure, but I would guess the internal routines use the Newton-Raphson (or variant thereof) method of finding the roots).

$$Det. = \frac{50s^3 + 65s^2 + 74s + 22}{s^2} = \frac{50(s + .391)\left((s + .455)^2 + .958^2\right)}{s^2}$$

Assume that V_{out} is taken across the inductor, in that case

$$V_{out} = i_3(s)sL$$

$$i_3(s) = \frac{\begin{bmatrix} \left(5 + \frac{1}{s}\right) & -\frac{1}{s} & V_{in}(s) \\ -\frac{1}{s} & \left(5 + \frac{3}{s}\right) & 0 \\ 0 & -\frac{2}{s} & 0 \end{bmatrix}}{Det.}$$

$$i_3(s) = \frac{2V_{in}}{50(s + .391)\left((s + .455)^2 + .958^2\right)}$$

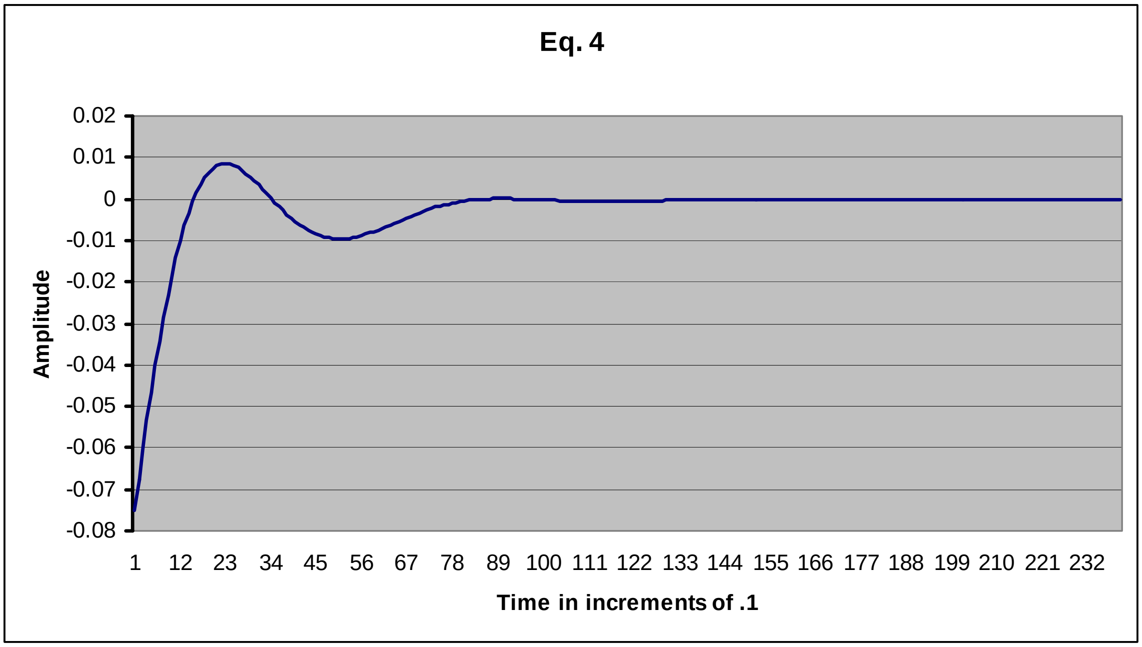
and

$$V_{out} = \frac{V_{in}4s}{50(s + .391)\left((s + .455)^2 + .958^2\right)} = \frac{V_{in}.08s}{(s + .391)\left((s + .455)^2 + .958^2\right)}$$

$$\frac{V_{out}}{V_{in}} \approx \frac{(.0923 * .958)\angle 21.6^o}{(s + .455)^2 + .958^2} - \frac{.034}{s + .391}$$

The impulse response is

$$f(t) = .0923e^{-.455t}\sin(.958t + 21.6^o) - .034e^{-.391t} \qquad \leftarrow \text{ Eq. 4}$$



Suppose the circuit is driven at a frequency near resonance

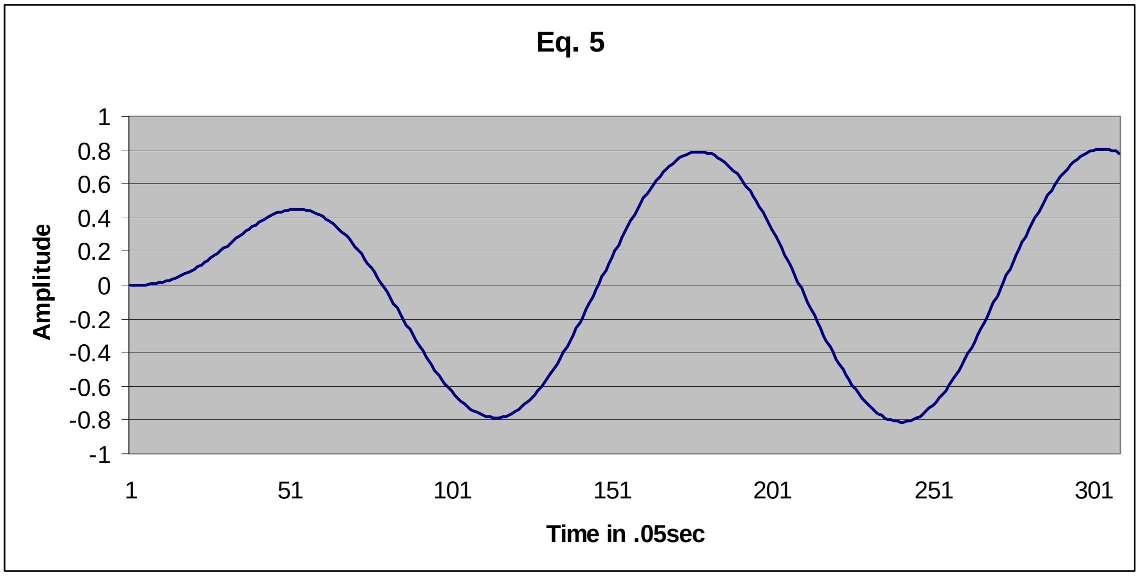
$$V_{in} = \frac{10}{s^2 + 1}$$

-
then

$$V_{out} = \frac{.8s}{(s+.391)\left((s+.455)^2 + .958^2\right)(s^2+1)}$$

$$V_{out} \approx \frac{-.294}{s+.391} + \frac{1.004 * .958\angle 93.2^o}{(s+.455)^2 + .958^2} + \frac{.81\angle -61^o}{s^2+1}$$

$$V_{out}(t) \approx .81\sin(t - 61^o) + 1.004e^{-.455t}\sin(.958t + 93.2^o) - .294e^{-.391t} \quad \leftarrow \text{Eq. 5}$$



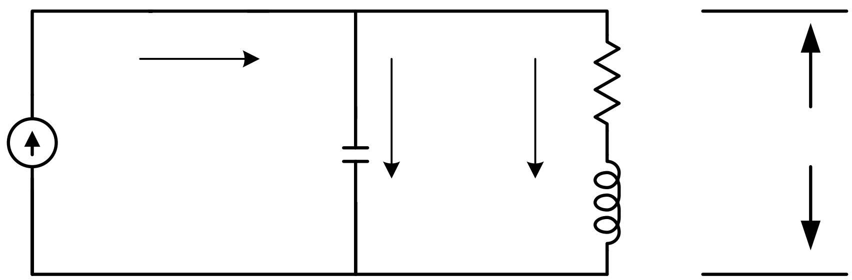
A Very Good Tank Circuit
or
The Strange Case of a Pulsed Driver

Suppose we wish to build a high quality tank circuit for frequency determining purposes, and further suppose that frequency is 1 megahertz (6.28 megarads/s). We know that $\pm j\sqrt{1/LC}$ is the resonant frequency (in rads/s) when $R = 0$ (an impossible condition in practical reality, but frequently used for pedagogical or rough estimation purposes) and that the frequency of oscillation (in rads/s) is $\pm j\sqrt{1/LC - \left(R/2L\right)^2}$ when $R \neq 0$ (see appendices D & E).

Further suppose we build the circuit as a parallel tank with no physical resistor in the circuit. We will model it as if the DC resistivity of the inductor and component leads is small; on the order of 10^{-1} ohms. Some design considerations will surface. In addition the drive will be a repetitious pulse (the analysis of the response will require some thought to reconcile the math to reality).

A drive of the nature described above possesses a spectrum of frequencies, however for illustrative purposes we will ignore the harmonic content at present. The development of the spectrum is a subject for Fourier analysis; we will hold that effort in abeyance for the purposes of this module. Just bear in mind that the actual output will contain a range of frequencies the majority of which will be attenuated by the filtering properties of the tank.

Choosing the lumped value of .1 ohms of lead resistance and .2 ohms of DC inductor resistance, and further letting $C = 10\,\text{pf}$; then $L \approx 2.533\,\text{mh}$, verify that this combination yields an $\omega_o \approx 6280000$ rads/s and that $|X_L| \approx |X_C|$. Question: what happens to the resonant frequency if the capacitor and the inductor values vary by $\pm 10\%$ in production? Does this condition present an envelope of frequencies? How could this be rectified to meet a spec of $\omega_o = 6280000 \pm 1\%$?



The above circuit has two loop currents which we will call i_C & i_L; and since

$$i_t = i_C + i_L$$

$$i_t = V_{out}\left(10^{-11}s + \frac{1}{.3 + .002533s}\right)$$

$$i_t = V_{out}\left(\frac{2.533X10^{-14}s^2 + 3X10^{-12}s + 1}{.002533(s + 78.96)}\right) = \left(\frac{10^{-11}(s^2 + 79s + 3.95X10^{13}}{(s + 78.96)}\right)$$

$$\frac{V_{out}}{i_t} \approx \frac{10^{11}(s+79)}{(s^2 + 79s + 3.95X10^{13})} \approx \frac{10^{11}(s+39.5)}{(s+39.5)^2 + (6.28X10^6)^2} + \frac{(6.28X10^6)(6.28X10^6)}{(s+39.5)^2 + (6.28X10^6)^2}$$

Finding the impulse response

$$V_{out}(t) \approx 1X10^{11}e^{-39.5t}\sin(6.28X10^6 t + 89.99^o)$$

Obviously a magnitude of 10^{11} volts is unsupportable in a small signal circuit. Like a 1910 transmitter there would be "arcs and sparks". There is nothing wrong with the math; the improbable result is a function of a practical impossibility.

$\delta(t)$ is an intellectual construct suitable for instructional purposes. A practical approximation of $\delta(t)$ is a finite amplitude pulse of a few nano seconds duration, that also possesses some finite rise and fall time. While the impulse response shown above yields a "peek under the tent" it does not yield an undistorted mirror of amplitude reality.

The transfer function is, as already developed

$$\frac{V_{out}}{i_t} = \frac{sL + R}{LC(s^2 + \frac{R}{L}s + \frac{1}{LC})} = \frac{s + \frac{R}{L}}{C(s^2 + \frac{R}{L}s + \frac{1}{LC})}$$

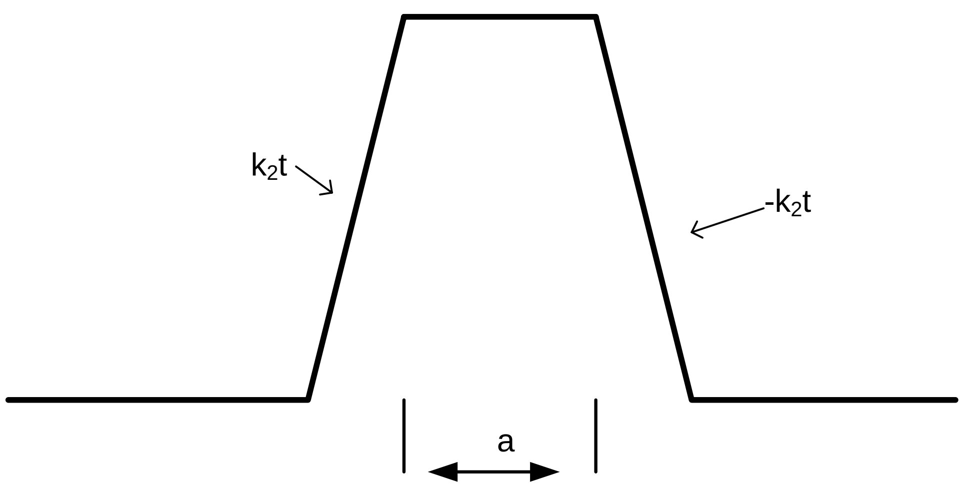
re-arranging

$$V_{out} = \left(\frac{s + \frac{R}{L}}{C(s^2 + \frac{R}{L}s + \frac{1}{LC})} \right) i_t$$

If we ignore the rise and fall times, and model $i_t = k_1(u(t) - u(t-a))$, where a is of a few nano seconds duration and k_1 is the amplitude, we get much closer to a practical result. We get even closer if the rise and fall times are included. The entire expression for such a driver might be

$$i_t = k_2 \{ (tu(t) - tu(t-\beta)) - tu(t-\chi) + tu(t-\varepsilon) \}$$

Where $\beta < \chi < \varepsilon$ and $\chi - \beta = a$, all on the order of a few nano seconds. The constant k_2 modifies the slope of t.



There are trade-offs in either scenario in the form of an unwanted frequency spectrum. and also perhaps in the form of more filtering by adding an additional tank or tanks. Adding additional tanks is not necessarily undesirable, as that narrows the bandwidth and provides increased selectivity (another topic for a later discussion, but not in this module). As always in reality we deal in trade-offs and look for the best compromise. One of the trade-offs is that even though the above model might more accurately reflect the actual driving pulse, it's mathematical implementation fails to reflect the reality of the output. For the sake of modeling and illustration and simplicity the following is an example worked for a driver consisting of a **step of 1mA @ 10nsec** duration. .

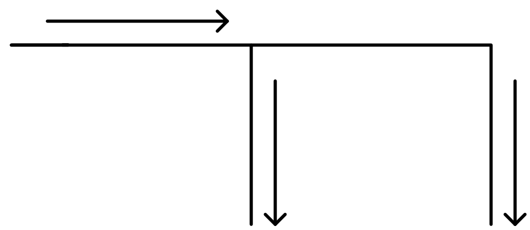
$$V_{out} = \left(\frac{\left(\frac{.001}{C}\right)\left(s+\frac{R}{L}\right)}{s(s^2+\frac{R}{L}s+\frac{1}{LC})} \right) - \left(\frac{\left(\frac{.001}{C}\right)\left(s+\frac{R}{L}\right)e^{-10^{-8}s}}{s(s^2+\frac{R}{L}s+\frac{1}{LC})} \right)$$

When transient response is the issue we should work only with the first expression to the right of the equals sign as the second will be a mirror image, albeit with a sign change and a time offset. That's the rub, it serves to cancel the output quite possibly before the circuit has had time to settle. The reasons for that is that the math model fails to follow reality in this case. Essentially we are treating a $\delta(t)$ as a finite amplitude pulse with finite duration to limit the predicted output amplitude to the neighborhood of reality. The above model predicts an output stop coincident with a driver stop, as the driver is modeled as a step. So, knowing that the tank will continue to oscillate until exhausted by the time constant rule of 6 (transient response), treat the driver as a mathematical impulse and ignore the driver duration WHEN it is much shorter than 6 time constants. Of course if the duration of the driver is greater than 6 time constants then model it as we did above. When the driver a time of existence much smaller than 6 time constants the LaPlace transform method does not handle that as well as other methods.

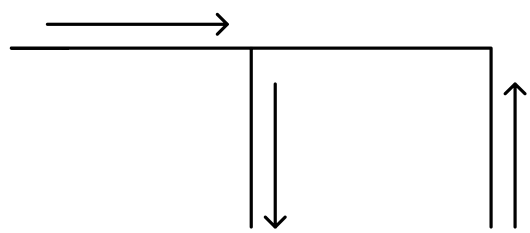
Why bother with modeling the driver as a short duration step at all? The physics of a tank are such that when current flow is initiated, it creates a back and forth transfer of charge between the inductor and the capacitor, much like the swing of a pendulum. Initially both the inductor and capacitor are uncharged. When current flow is initiated, the inductor resists the change, but the capacitor charges. Eventually the rate of change of charge in the capacitor diminishes as the inductor begins to charge. When the driver ceases, the inductor resists the change and continues to charge at the expense of the capacitor. Eventually the capacitor is discharged and current flow momentarily ceases. The inductor's magnetic field collapses at that point, recharging the capacitor in the opposite polarity. The process continues, back and forth transfer of charge (oscillation) until $i^2 r$ loses exhaust the circuit (expressed as a time constant). The driver **duration** determines the amount of charge available to create current flow, as $V_C = \frac{1}{C}\int i_c dt$; of course the limits on the integral are the duration of the driving step. In this case $V_C = 1V$.

The sum total of all of this discussion is that if you choose to approximate an actual small duration driver as an amplitude limited $\delta(t)$, your output prediction will be much closer to the actual case.

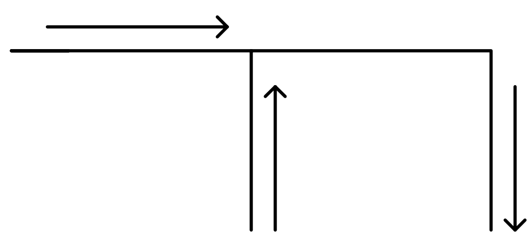
Another, and in many ways more satisfying, way of looking at tank currents is to visualize the various flow conditions using Kirchhoff's current law. The only assumption we need make is that when current is not flowing from the driver (bear in mind that we are modeling with a current driver), little or no current can flow into the driver; i.e., it behaves as an open.



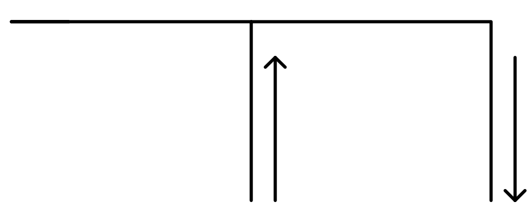
Case 1: Initial current flow; reactive components not yet charged.



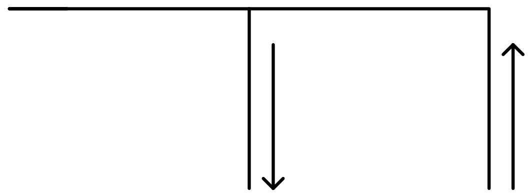
Case 2: Current flowing from both the driver and the inductor.



Case 3: Current flowing from both the driver and the capacitor.



Case 4: Driver off; capacitor discharging into the inductor.



Case 5: Driver off; inductor discharging in to the capacitor.

As the above are the only possible cases of current flow, it is fairly easy to visualize what is going on with the driver either on or off. The majority of the time Cases 4 & 5 predominate as the driver is generally on only for a small duty cycle.

V_{out} can also be re-written as

$$V_{out} = \left(\frac{s + \frac{R}{L}}{C(s^2 + \frac{R}{L}s + \frac{1}{LC})} \right) \left(\frac{.001}{s} \right) \left(1 - e^{-10^{-8}s} \right)$$

with values

$$V_{out} = \frac{\left(\frac{.001}{10^{-11}} \right) \left(s + \frac{.3}{.002533} \right)}{s\left((s+39.5)^2 + (6.28X10^6)^2 \right)} = \frac{A}{s} + \frac{B}{(s+39.5)^2 + (6.28x10^6)^2}$$

$$V_{out} = \frac{\left(\frac{.001}{10^{-11}} \right) \left(s + \frac{.3}{.002533} \right)}{s\left((s+39.5)^2 + (6.28X10^6)^2 \right)} \approx \frac{.0003}{s} + \frac{16 * (6.28X10^6)^2}{(s+39.5)^2 + (6.28x10^6)^2}$$

As Bell[1] would say "FAPP" (For All Practical Purposes)

$$V_{out}(t) \approx 16e^{-39.5t} \sin(6.28X10^6 t)$$

Our math predicts a magnitude of 16; that is because the input was modeled by manipulating unit steps. But the input is a 10ns wide pulse that does not have the time available to completely charge the circuit (if the circuit completely discharges in 6 time constants. it will charge in 6 time constants; recall that $V_C = \frac{1}{C}\int idt \approx 1V$). A Fourier transform approach would be a better approach in generating a model of the response; but Fourier transforms yield a frequency spectrum and herein we concerned with the time response.

[1] John S. Bell, 1928-1990, Physicist extraordinaire

Also an instantaneous polarity reversal at 10^{-8} seconds is expected from the model because we have assumed an ideal driver with instantaneous rise and fall times. Instantaneous "anythings" are generally not possible. Since $V_L = L\frac{di}{dt}$ and $\frac{di}{dt} = \infty$ for an instantaneous change (a discontinuity, i.e., the driver has two different simultaneous values) the model fails. That must be kept in mind to make the final result meaningful. That is another reason why we fall back on FAPP. FAPP is often an engineer's best friend.

A little bit of driving current yields an oftimes magnificent output, sometimes more than the circuit is intended for and additional circuitry is required to keep things sane. Nevertheless, even a driver that is a timed and sequenced unit step does not yield an undistorted picture of reality, but it is much closer than the impulse response.

A look at the time constant indicates that the output has decayed to 36.7% of max at about 25ms. Suppose it is desired that the circuit never be allowed to decay below 90% of max. What is the interval needed between driving impulses? So

$$e^{-39.5\tau} = .9$$

$$-39.5\tau \ln e = \ln .9$$

$$\tau \approx 2.7ms$$

so the circuit is pulsed every 2.7ms.

In the case above, the driver is on for 10ns and cycle time is 2.7ms; so the duty cycle is about .0004%. That is a pretty decent tank.

In the above example, remember the time constant is actually $\frac{1}{39.5}$, as the units on $\frac{39.5}{1}$ MUST be t^{-1}. 39.5 represents $\frac{R}{2L}$, therefore the time constant is $\frac{2L}{R}$ - does this expression have units of t? Yes, it does.

Not developed in this discussion is the bandwidth of the tank and spectrum of the output. Both topics are of critical interest to the analyst and to the designer. Neither can be ignored in practice. They have been tacitly ignored in this discussion as the intention here is to focus on the time response. Discussions and development of tank bandwidth and ability to discriminate against unwanted frequencies will be delayed until a later module.

Nevertheless, it is truly the frequency content of the driving function that causes sympathetic oscillation in the circuit. That aspect has been totally ignored in this discussion as it is not needed in the development of a simple time response. But as we shall see in subsequent modules frequency is a major player, often the 800 pound Gorilla, in circuit stability/instability considerations.

Transform	$f(t)$	$F(s)$
1	K	$\frac{K}{s}$
2	$Ke^{-\sigma t}$	$\frac{K}{s+\sigma}$
3	$K\sin(\omega t)$	$\frac{K\omega}{s^2+\omega^2}$
4	$K\cos(\omega t)$	$\frac{Ks}{s^2+\omega^2}$
5	$Ke^{-\sigma t}\sin(\omega t)$	$\frac{K\omega}{(s+\sigma)^2+\omega^2}$
6	$Ke^{-\sigma t}\cos(\omega t)$	$\frac{K(s+\sigma)}{\left((s+\sigma)^2+\omega^2\right)}$
7	$\delta(t)$	1
7a*	$K\delta(t)$	K
8	$Ku(t-a)$	$\frac{Ke^{-as}}{s}$
9	$f'(t)$	$sF(s)-f(0)$
10	$\int f(t)dt$	$\frac{F(s)}{s}$
11	$af(t)+bg(t)$	$aF(s)+bG(s)$
12	t	$\frac{1}{s^2}$
13	te^{-at}	$\frac{1}{(s+a)^2}$

Table 1

** K is preserved for practical circuit reasons, not for theoretical reasons as* $K*\infty$ *is approximately equal to* ∞

It is **very important** to understand that to be able transform any $F(s)$ to an $f(t)$, $F(s)$ **must** be reduced to one of the forms so far developed. If it is not in one of these forms it cannot be operated on until it is. Study the right hand side forms, they identify the left hand side.

Appendix A
Cramer's Rule
Refresher

This appendix is not intended as a tutorial, but rather as a refresher for those that want a reminder of how the process proceeds.

Suppose there is a system of equations representing a simple two loop circuit containing two unknowns and two equations, such as

$$a_1 i_1 + b_1 i_2 = V_{in}$$
$$a_2 i_1 + b_2 i_2 = 0$$

In the above case i_1 & i_2 are the unknowns and everything else is known.

Then using the notation of Algebra we can re-write these equations as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} V_{in} \\ 0 \end{bmatrix}$$

In order to solve for the two unknowns, **first** the determinant (Det.) is found by multiplying the elements of left to right diagonal, and then subtracting the multiplication of elements on the right to left diagonal.

Det. = $a_1 b_2 - b_1 a_2$

Next, to find i_1, the rightmost column containing V_{in} & 0 is substituted for the column containing a_1 & a_2.

$$\begin{bmatrix} V_{in} & b_1 \\ 0 & b_2 \end{bmatrix}$$

Then find the determinant of that new matrix.

$$V_{in} b_1 - b_2 0$$

Next divide by the Det., the result is the value of i_1. So

$$i_1 = \frac{V_{in} b_1 - b_2 0}{a_1 b_2 - b_1 a_2}$$

The term $b_2 0$ is zero, and is included only for completeness as the driver in the second loop is not always zero.

Finally solve for i_2 by substituting V_{in} & 0 for the column containing b_1 & b_2.

$$\begin{bmatrix} a_1 & V_{in} \\ a_2 & 0 \end{bmatrix}$$

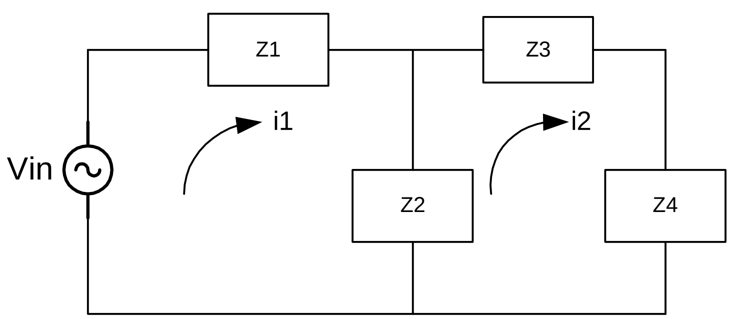
and divide by Det.

$$i_2 = \frac{a_1 0 - V_{in} a_2}{a_1 b_2 - b_1 a_2}$$

Please bear in mind that the second loop may have a driver and therefore the left hand side of the describing equation will not be zero. Also, there is no constraint on the matrix elements to be real, in fact in actual circuitry they are more than often complex. Also because the rules of LaPlace Transform pairs allows addition, the whole set of equations may be written in the 's' domain.

Optional: For a short discussion of why this technique works, see appendix B.

An example;



From the above,

$$V_{in} = i_1(z_1 + z_2) - i_2 z_2$$
$$0 = -i_1 z_2 + i_2(z_2 + z_3 + z_4)$$

The determinant is $(z_1 + z_2)(z_2 + z_3 + z_4) - z_2^2$ or

Det. $= z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4$

Then $i_1 = \dfrac{V_{in}(z_2 + z_3 + z_4) + z_2 0}{Det.}$

and $i_2 = \dfrac{i_1 0 + V_{in}(z_2)}{Det.}$

Assume: $V_{in} = 10$, $z_1 = z_4 = 10$, $z_2 = z_3 = 5$

Then Det. is $\begin{bmatrix} 15 & -5 \\ -5 & 20 \end{bmatrix} = 275$

and

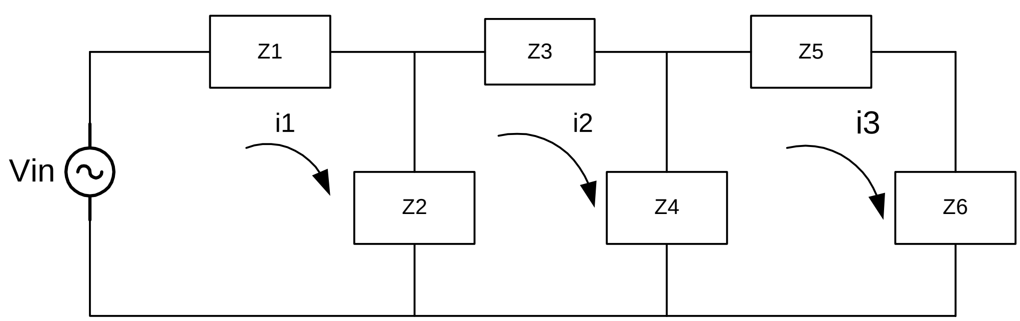
$$i_1 = \frac{\begin{bmatrix} 10 & -5 \\ 0 & 20 \end{bmatrix}}{275} = \frac{200}{275} \cong .73$$

and

$$i_2 = \frac{\begin{bmatrix} 15 & 10 \\ -5 & 0 \end{bmatrix}}{275} = \frac{50}{275} \cong .18$$

Suppose $V_{out} = i_2 z_4$, then $V_{out} = 1.8V$

For a three loop circuit, there will be three equations and three unknowns.



The procedure for any matrix greater than a 2x2 as in the above example, is extended and modified slightly. In the case of the three loop circuit there a three columns and three rows;

$$\begin{aligned} V_{in} &= i_1(z_1 + z_2) - i_2(z_2) + i_3(0) \\ 0 &= -i_1 z_2 + i_2(z_2 + z_3 + z_4) - i_3 z_4 \\ 0 &= i_1(0) - i_2 z_4 + i_3(z_4 + z_5 + z_6) \end{aligned}$$

The determinant matrix will be

$$\begin{bmatrix} (z_1+z_2) & -z_2 & 0 \\ -z_2 & (z_2+z_3+z_4) & -z_4 \\ 0 & -z_4 & (z_4+z_5+z_6) \end{bmatrix}$$

At this point a modification occurs. There are three columns, and therefore there must be three terms for both the left and right hand diagonals. A frequent crutch that works is to merely copy columns 1 & 2 to the right of the matrix; that yields three complete diagonals in each direction. To extend the rule, an nxn matrix requires n diagonals in each direction.

$$\begin{bmatrix} (z_1+z_2) & -z_2 & 0 \\ -z_2 & (z_2+z_3+z_4) & -z_4 \\ 0 & -z_4 & (z_4+z_5+z_6) \end{bmatrix}\begin{matrix} (z_1+z_2) & -z_2 \\ -z_2 & (z_2+z_3+z_4) \\ 0 & -z_4 \end{matrix}$$

the determinant then is

$$\begin{aligned}(z_1+z_2)(z_2+z_3+z_4)(z_4+z_5+z_6)+(-z_2)(-z_4)(0)+(0)(-z_2)(-z_4) \\ -(0)(z_2+z_3+z_4)(0)-(z_1+z_2)(z_4^2)-(z_2^2)(z_4+z_5+z_6)\end{aligned}$$

As an example, suppose that three equations in three unknown are

$$\begin{aligned} 20 &= 15i_1 - 5i._2 \\ 0 &= -5i_1 + 20i_2 - 5i_3 \\ -5 &= -5i_2 + 15i_3 \end{aligned}$$

The matrices are

$$\begin{bmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 15 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ -5 \end{bmatrix}$$

The determinant is

$$\begin{bmatrix} 15 & -5 & 0 \\ -5 & 20 & -5 \\ 0 & -5 & 15 \end{bmatrix}\begin{matrix} 15 & -5 \\ -5 & 20 \\ 0 & -5 \end{matrix}$$

$$(15)(20)(15)+(-5)(-5)(0)+(0)(-5)(-5)-(0)(20)(0)-(15)(-5)(-5)-(-5)(-5)(15)=3750$$

$$i_1 = \frac{\begin{bmatrix} 20 & -5 & 0 \\ 0 & 20 & -5 \\ -5 & -5 & 15 \end{bmatrix}}{3750} = 1.433$$

$$i_2 = \frac{\begin{bmatrix} 15 & 20 & 0 \\ -5 & 0 & -5 \\ 0 & -5 & 15 \end{bmatrix}}{3750} = .3$$

$$i_3 = \frac{\begin{bmatrix} 15 & -5 & 20 \\ -5 & 20 & 0 \\ 0 & -5 & -5 \end{bmatrix}}{3750} = -.233$$

The minus sign on i_3 merely means that it is flowing in a direction opposite to the other two.

For matrices greater than three rows by three columns (3x3), the labor goes up significantly, and the use of a calculator such as an HP49 or a computer program similar to MATLAB is very helpful. However for those that enjoy the labor the following rules are offered:

The determinant is the algebraic sum of all the possible products where:

a. each product has factors of one element, and only one, from each row and column
b. a plus sign is assigned to each product if the number of column inversions is even, 0 inversions being defined as even. A minus sign is assigned to a product that has an odd number of column inversion.

An inversion, by illustration, is that if the natural order of counting is 1234 and a product is formed from columns 1423 then it contains two inversions; to change 1423 to 1234, the 4 must move two places to the right. 4321 has six inversions as the 4 moves three places, the 3 moves two places and the 2 moves one to create 1234.

These rules are simply the procedure used for a 3x3 extended to an nxn.

There are other, equally valid techniques from our slide-rule days, such as Gaussian Elimination, Matrix Inversion and the use of Minors, but all in all once you understand the foundations of the process the use of a good calculator is incredibly labor and error saving. Of course it is the understanding of these techniques that form the foundations for the programming in the calculator's ROM library.

Appendix B
Foundations

Suppose

$$Ax + By = P$$
$$Cx + Dy = Q$$

then $By = P - Ax$ and $y = \frac{P - Ax}{B}$

then $Cx + D\left(\frac{P - Ax}{B}\right) = Q$ and becomes $CBx + DP - ADx = BQ$

which, in turn becomes $x(CB - AD) = BQ - DP$

or $x = \frac{PD - BQ}{AD - BC}$

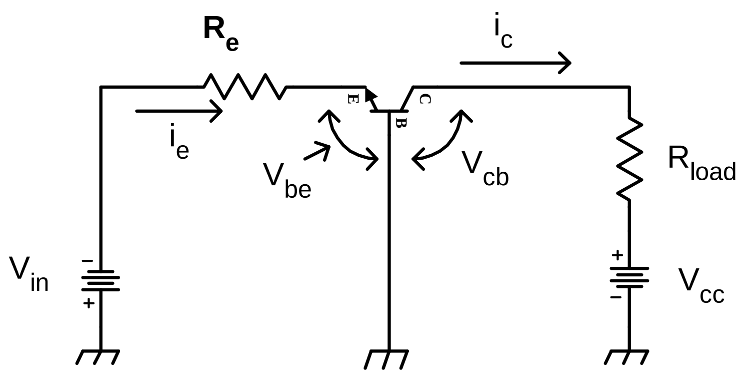
Using Cramer's rule the matrices for the two initial equations are

$$x = \frac{\begin{bmatrix} P & B \\ Q & D \end{bmatrix}}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} = \frac{PD - BQ}{AD - BC}$$

For those with infinite stamina this procedure can be extended to any number of equations that possess the same number in unknowns. But the author, being a member of Lazyhood Incorporated, uses mechanical means once the theory is established.

Appendix C
Common Base Amplifier

Consider the following circuit



This circuit consists of an NPN transistor, forward biased emitter to base (V_{be}) and reverse biased collector to base (V_{be}). Typically V_{be} is on the order of approximately .7 volts, so the current through R_e, is

$$I_e \approx \frac{V_{in} - .7}{R_e}$$

The physics of the transistor are such that the current through the collector and hence through R_{load} is always .98-.99 I_e (true within the manufacturers operating characteristics range for the particular transistor type). Therefore adjusting V_{in} adjusts I_e, which in turn controls the load current. In short

$$I_c \approx .99 I_e$$

Obviously this makes the current through the load utterly dependent on I_e, which in turn is dependent upon the values chosen for $R_e \,\&\, V_{in}$.

The above is a very primitive version of a common base amplifier, and design considerations of coupling, impedance, bandwidth, emitter resistance, etc., have been utterly ignored so as to focus on the current generator effect at the collector.

The most common point of confusion is that if the collector current is 99% of the emitter current, what has happened to ohm's law in the collector to base loop. Nothing actually. Kirchhoff's voltage equation for that loop is

$$V_{cc} \approx I_e R_{load} + V_{cb}$$

As the transistor is active, and is constrained by it's physics, V_{cb} adjusts to accommodate the voltage law.

Appendix D

Resonance is defined to occur when $|X_L| = |X_C|$. For a series circuit the total impedance is

$$R_t + jX_L - jX_C$$

It is clear at resonance the impedance is at minimum and depends only upon R_t.

In a parallel circuit the above sum is the denominator of the expression of total impedance. Again it is clear that it is a minimum at resonance. A minimum denominator creates a maximum impedance.

As a rule of thumb at resonance: a) a series circuit approaches a short, and b) a parallel circuit approaches an open.

Since $X_L = j2\pi fL$, and $X_C = \frac{-j}{2\pi fC}$, at resonance

$$|2\pi f_o L| = \left|\frac{1}{2\pi f_o C}\right|$$

since $2\pi f_o = \omega_o$ we can re-write as follows

$$\omega_o^2 = \frac{1}{LC}$$

an finally

$$\omega_o = \sqrt{\frac{1}{LC}}$$

Appendix E

A quadratic has only two possible sets of roots; a. both real (may be equal or unequal) or b. a complex pair. Consider the quadratic

$$s^2 + \frac{R}{L}s + \frac{1}{LC}$$

this roots are

$$-\frac{R}{2L} \pm \frac{\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

which leads to

$$-\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

when the discriminant is ≥ 0, the roots are real, and when the discriminant < 0 the roots are complex and may be re-written conveniently as

$$-\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$